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# Optimal mean-square performance for networked control systems with unreliable communication

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**Optimal mean-square performance for networked control systems with unreliable  
communication**

by

Matthew Christopher Rich

A dissertation submitted to the graduate faculty  
in partial fulfillment of the requirements for the degree of  
DOCTOR OF PHILOSOPHY

Major: Electrical Engineering (Systems and Controls)

Program of Study Committee:  
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## ABSTRACT

In this work we study mean-square (MS) stability and mean-square (MS) performance for discrete-time, finite-dimensional linear time-varying systems with dynamics subject to i.i.d. random variation. We do so primarily in the context of networked control systems (NCS) where the network communication channels are unreliable and modeled as multiplicative stochastic uncertainties, e.g. wireless links subject to packet dropouts and modeled as Bernoulli processes.

We first focus on the analysis problem in general. We derive a convex feasibility problem and associated convex optimization problem which can be used to determine the MS stability and MS performance respectively of a given system. Since this analysis theory is derived in terms of the feasibility of and optimization of a linear cost subject to linear matrix inequalities (LMIs), it serves as the foundation from which a solution methodology for numerous controller synthesis problems can be derived.

Next we formulate the main synthesis problem we consider in this work: a networked control system where both the sensor measurements for the plant(s) and the commands from the controller are transmitted via unreliable communication channels. We treat the unreliable communication links as i.i.d. random processes. We assume that the plant(s) and links are subject to additive exogenous noise, and that we have access to a reliable but delayed acknowledgment of whether or not the controller commands were received by the plant(s) on the previous time step. Finally we restrict the controller to be finite-dimensional, linear, have no structural dependence on the particular path history of the random processes, and scale in size and complexity linearly with the number of random channels.

We then show that this synthesis problem has a MS stabilizing solution if and only if two simpler convex problems have MS stabilizing solutions, and moreover that the optimal MS performance solution to this synthesis problem if it exists can be obtained by solving a sequence of these simpler convex problems. Additionally, we show that the overall optimal MS

performance cost is the sum of two components which can be determined from the solutions to the special problems. That is, we derive a separation principle for our problem analogous to the classical  $\mathcal{H}_2$  synthesis.



## CHAPTER 1. INTRODUCTION

### 1.1 Networked Control Systems

Networked control systems (NCSs) have been a major area of research in both industry and academia for decades, with the spread of wireless technologies increasing both the interest in and deployment of networked control systems tremendously [1–3]. Overall the main distinguishing characteristic of a NCS is physically distributed components using possibly non-ideal network links for some or all of their communication. An example is shown in Figure 1.1, where one or more quadrotors (generally termed the plant or plants) are controlled from a remotely located computer which transmits commands via an unreliable wireless link, and receives sensor data from remotely located sensors via another unreliable network link.

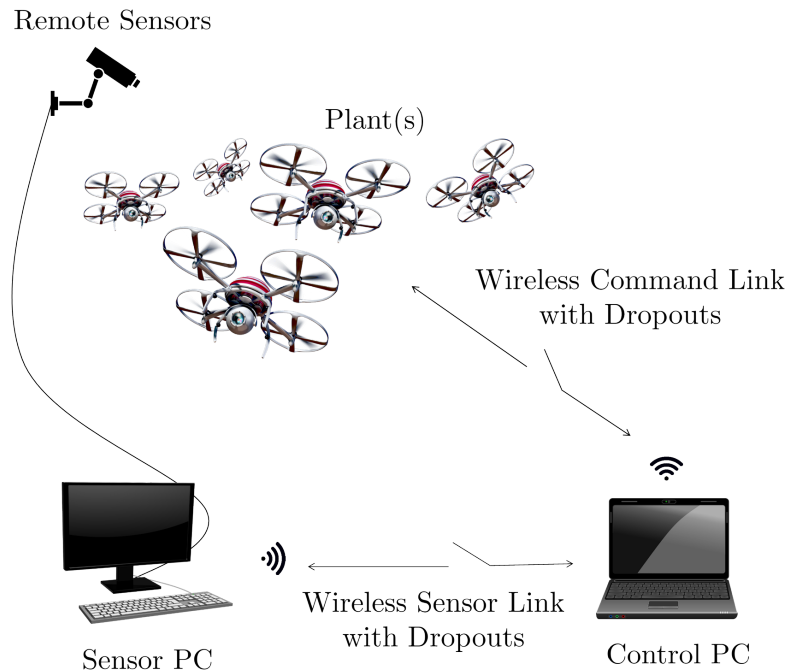


Figure 1.1: Example of a Networked Control System

Three basic high-level NCS architectures are shown in Figure 1.2, where  $P$  and  $P_i$  represent the plant(s), i.e. the system(s) to be controlled,  $K$  and  $K_i$  represent the controller(s), and  $N$  represents the network interconnections. In Figure 1.2a the plant measurements  $y_p$  (which may include both local and remote sensor readings), are transmitted through the network and received by the controller as  $y$ , which may differ from  $y_p$  at a given time due to some network effects. Similarly, the controller generated commands  $u$  arrive at the plant as the possibly altered signal  $u_p$ . Figure 1.2b generalizes this idea to multiple plants (often called a swarm) controlled by a single controller. Figure 1.2c represents a distributed control scenario where multiple plants with local controllers communicate via a network. Generally, an NCS may consist of combinations of these or other possible architectures not shown here.

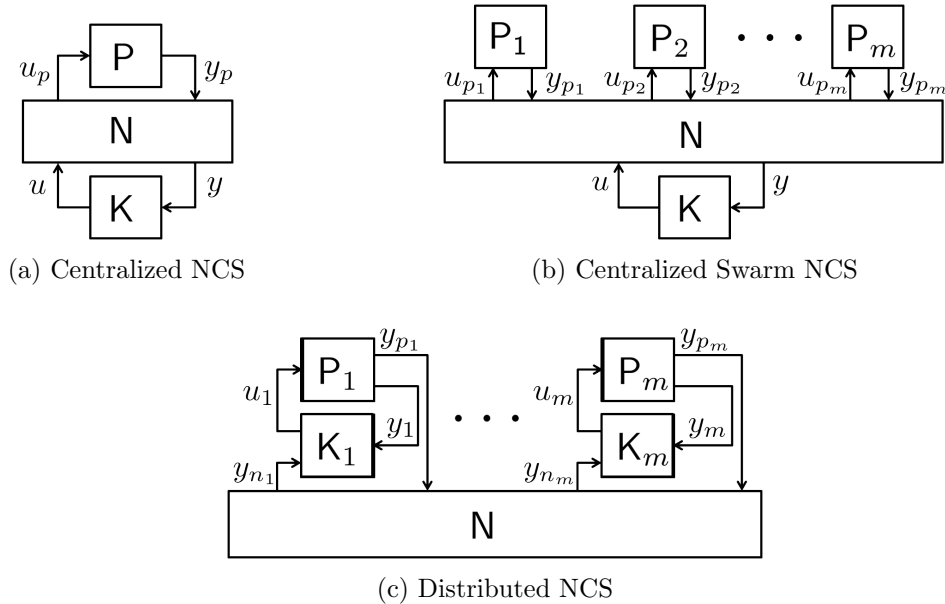


Figure 1.2: Basic Networked Control System Configurations

We point out that from a controller synthesis perspective, Figure 1.2a and 1.2b are essentially equivalent since multiple plants can be modeled as a single, larger system with structure and a single central controller designed. However for the NCS in Figure 1.2c, the equivalent controller is distributed and therefore has internal structural constraints. These additional constraints mean we cannot approach controller synthesis the same way as the centralized, unstructured control setups. In this work we will only consider scenarios which can be represented as in Figure 1.2a or 1.2b, i.e. centralized control designs.

A small sample of some of the current areas of research and application of networked control systems includes autonomous and/or remotely controlled aerospace and automotive vehicles and vehicle networks, exploration of hazardous environments either via teleoperation or autonomous navigation, optimization of transportation and power distribution systems, various applications of cooperative and/or competitive robotics, performance of emergency surgery or other medical procedures via teleoperation, and general distributed computation and optimization. As previously mentioned NCS research has been quite popular for some time, and the body of related literature is consequently both immense and diverse. We refer the reader interested in a survey of recent NCS literature to [1–7].

In many recent works dealing with NCS, like this one, the network communication is assumed to be subject to some form of uncertainty. Often the uncertainty has been assumed to be stochastic. In this context for example [8–15] have studied Kalman filtering and estimation problems, [16–22] LQG control synthesis, [23–32] limitations of mean-square (MS) stability, and [33–37] MS performance. Markov jump linear systems (MJLS) approaches [38–45] have been applied to NCS problems under various assumptions on the stochastic nature of the network. Distributed consensus and least squares problems with unreliable network interconnections have been studied in [46–51]. Event triggered systems approaches have been applied to NCS problems with network uncertainty in [52–55]. Recently there has also been a growing interest in cyber-security related to NCS [56–59], where the network disruption is no longer random but due to purposeful attack.

## 1.2 Summary

In this work we study stability and performance analysis and optimal performance controller synthesis for a class of linear stochastic systems, primarily from a networked control systems (NCS) perspective. More specifically, we investigate mean-square (MS) stability and mean-square (MS) performance properties for multiple-input multiple-output (MIMO), discrete-time, finite-dimensional linear systems whose dynamics are subject to random time-variation and additive exogenous noise. We assume the variation is described by random processes which are

independent and identically distributed (i.i.d.) in time. For example, networked control system where the communication links are wireless and subject to random packet losses which can be modeled as Bernoulli processes.

Our development here builds directly upon and/or generalizes results in a number of previous works [23,27,28,30,33–37,60]. The general theory on linear matrix inequalities (LMIs) which forms the foundation of the MS stability and later MS performance results in [23,27,28,30,33–37] and this work was derived in [60]. The fundamental framework we use for NCS problems was defined in [23]. In [33,34] optimal MS performance controller synthesis for a single-input single-output (SISO) systems was addressed via nonconvex search. In [30] the problem of convex MS stabilizing controller synthesis for SISO systems with both actuator and sensor link fading was studied. In [27] we derived a convex means to solve the MS stabilizing controller synthesis problem for MIMO systems with actuator link dropouts, as well as a means to numerically analyze the necessary and sufficient limitations of MS stabilizability for such systems. In [28] we extended the core MS stabilizing controller synthesis results of [30] to the MIMO equivalent problem setup. In [35,36] we provide preliminary extensions of [27] to optimal MS performance.

In Chapter 2 we begin with a simple introductory example to motivate our approach to network control system analysis and control design. We summarize the framework we will use and provide a list of the standing assumptions which generally define the scope of the problems considered in this work. We end this chapter by formally defining the notions of stability and performance we apply to the systems we consider.

In Chapter 3 address MS performance analysis. First we show that the MS stability and MS performance properties of a given system can be determined using a matrix linear time-invariant system corresponding in structure to the mean closed loop. We use this result to derive a notion of MS stability and performance duality for the systems we consider as well as to derive a convex feasibility problem and associated convex optimization problem which can be used to determine the MS stability and MS performance respectively of any given system. These latter results are derived in terms of linear matrix inequalities (LMIs) and serve as the primary foundation for the synthesis theory later in this work.

In Chapter 4 we provide the formulation for the main control synthesis problem of this work: a networked control system where both sensor and actuator signals are transmitted through stochastic links. We show how we represent this problem in our framework and we define the assumptions and restrictions we impose on the synthesis.

In Chapter 5 we investigate a sequence of problems analogous to the classical special problems: Full Information, Disturbance Feedforward, Full Control, and Output Estimation [61]. We show that we can solve these problems convexly as well as derive numerous important properties of the solutions. As with the classical setting, these networked special problems will form the basis for the Youla parameterization of all MS stabilizing controllers and MS stable closed loops for the networked output feedback problem in Chapter 6, as well as the derivation of the separation principle.

In Chapter 6 we begin by slightly modifying the synthesis problem from Chapter 4. We then show that this problem has a MS stabilizing solution if and only if a pair of the special problems studied in Chapter 5 have a solution, and moreover that the optimal solution can be obtained by solving these problems in sequence. We then show that this solution can be adapted to provide the solution to the optimal MS performance synthesis problem in Section 4 as a special case.

Finally, in Chapter 7 we provide some numerical examples and in Appendices A-D we provide some additional background theory and the proofs for results in Chapters 3,5, and 6. Appendix E includes a reference for notation and acronyms throughout this work.

It is important to note that we will restrict our admissible control designs to those which do not depend on the path of the random channel states, which do not scale in size or complexity combinatorially with the number of random network states, and which allow one to calculate all control gains and optimal performance costs offline. These are fundamental contrasts with the problem formulations and subsequent controller designs in MJLS literature which scale combinatorially with the number of random processes, as well as with the LQG results where neither the optimal cost or controller can be computed offline due to the fact that the time-varying Kalman filter gain does not converge even in the infinite horizon case due to sample path dependence on the random channel states.

### 1.3 Notation

Although most of the notation we use in this work is relatively common or can be understood in context, we now provide a brief overview of some of the conventions and notation we will use. A more comprehensive reference is provided in Appendix E (page 89).

With very few exceptions we differentiate between various types of mathematical quantities by using distinct fonts. In this regard we use the default L<sup>A</sup>T<sub>E</sub>X math font for vectors, matrices, and random variables, e.g.  $v$ ,  $M$ , and  $\delta$ . For linear systems, we use the L<sup>A</sup>T<sub>E</sub>X Sans-serif math font, e.g.  $\mathbf{G}$  and  $\mathbf{K}$ . Operators which are represented using Roman letters are given using the standard upright L<sup>A</sup>T<sub>E</sub>X math font, e.g.  $\text{tr}(X)$  for the trace of a matrix  $X$ . Sets are denoted with the L<sup>A</sup>T<sub>E</sub>X bold math font, e.g.  $\mathbf{R}$  for the real numbers.

Some specific sets will often refer to are  $\mathbf{S}^n$ ,  $n \times n$  symmetric matrices,  $\mathbf{D}^n$ ,  $n \times n$  diagonal matrices, and  $\mathbf{B}^n$ , block diagonal matrices made up of  $p$  symmetric blocks of sizes  $\eta_i \times \eta_i$  defined in the  $p \times 1$  vector  $\eta$ . That is, any  $M \in \mathbf{B}^n$  is such that  $M = \text{diag}(M_1, M_2, \dots, M_p)$  where  $M_i \in \mathbf{S}^{\eta_i}$ . For each set, a + subscript denotes positive semidefinite, while a ++ denotes positive definite, e.g.  $\mathbf{S}_{++}^n$  is the set of  $n \times n$  symmetric positive definite matrices.

$I_n$  (or simply  $I$ ) denotes an  $n \times n$  identity matrix (or whatever size is appropriate in context).  $\mathbf{1}_{m \times n}$  is an  $m \times n$  matrix of ones, while  $\mathbf{1}$  is a vector of ones of whatever size appropriate in context.  $\mathbf{0}$  is always a matrix of zeros of whatever size is appropriate in context. The notation  $A \circ B$  represents the Hadamard or element-by-element product of two matrices (or vectors)  $A$  and  $B$  of the same dimensions. The notation  $A \succ 0$  or  $A \succeq 0$  is taken to mean the matrix  $A$  is symmetric and positive definite or positive semidefinite respectively.

We use  $F_\ell(\mathbf{G}, \mathbf{K})$  to represent the lower linear fractional transformation<sup>1</sup> (LFT) of some systems  $\mathbf{G}$  and  $\mathbf{K}$  of compatible dimensions. We will use a shorthand notation  $x^+$  for  $x(k+1)$  when providing a realization for a linear system and omit the time index  $k$ . For example, let a finite-dimensional linear time-invariant (FDLTI) system  $\mathbf{P}$  with input  $u$  and output  $y$

<sup>1</sup>Essentially feedback interconnection. See [62] or [61].

have a state space realization with internal state  $x$ , where  $x(k+1) = Ax(k) + Bu(k)$  and  $y(k) = Cx(k) + Du(k)$ . We will define this realization as in (1.1).

$$\mathbf{P} : \begin{bmatrix} x^+ \\ y \end{bmatrix} = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \begin{bmatrix} x \\ u \end{bmatrix} \quad (1.1)$$

Finally, we will denote the dual of an LTI system using the matrix transpose notation, e.g.  $\mathbf{P}^T$ .

Given that  $\mathbf{P}$  has a realization (1.1), the dual system has a realization (1.2).

$$\mathbf{P}^T : \begin{bmatrix} \tilde{x}^+ \\ \tilde{y} \end{bmatrix} = \left[ \begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array} \right] \begin{bmatrix} \tilde{x} \\ \tilde{u} \end{bmatrix} \quad (1.2)$$

## CHAPTER 2. PRELIMINARIES

### 2.1 Introductory Motivating Example

Consider the basic control loop shown in Figure 2.1 where  $P$  is the plant to be controlled,  $K$  is the controller,  $u(k)$  is the control command,  $y(k)$  is the measurement,  $n(k)$  is an exogenous disturbance, and  $r(k)$  is some regulated output of interest.

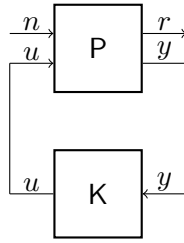


Figure 2.1: Basic Feedback Control Loop

If we consider the case where both the plant and controller to be finite-dimensional, linear time-invariant systems analyzing the stability of the overall closed loop boils down to simply checking the eigenvalues of a matrix. Specifically, in this scenario we can describe the overall closed loop using a state space realization

$$\begin{bmatrix} x^+ \\ r \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} x \\ n \end{bmatrix} \quad (2.1)$$

The closed loop is said to be stable if all the eigenvalues of  $\hat{A}$  have magnitude less than 1, i.e. if the matrix  $\hat{A}$  is Schur. This guarantees among other things that for zero input the system state  $x(k) \rightarrow 0$  as  $k \rightarrow \infty$  for any initial condition  $x(0)$ .



In contrast, consider the control loop shown in Figure 2.2a where  $y_P(k)$  is the measurement, and the controller  $K$  does not receive  $y_P(k)$  at each time  $k$  but rather  $y(k) = \xi(k)y_P(k)$ , where  $\xi(k)$  is a random process. Take for example a Bernoulli process where at each time  $k$  the probability that  $\xi(k) = 1$  is  $\mu$  and the probability that  $\xi(k) = 0$  is  $1 - \mu$ . This is a simple model for a network link subject to random dropout where the controller receives the measurement at time  $k$  with probability  $\mu$  and receives no update at time  $k$  with probability  $1 - \mu$ .

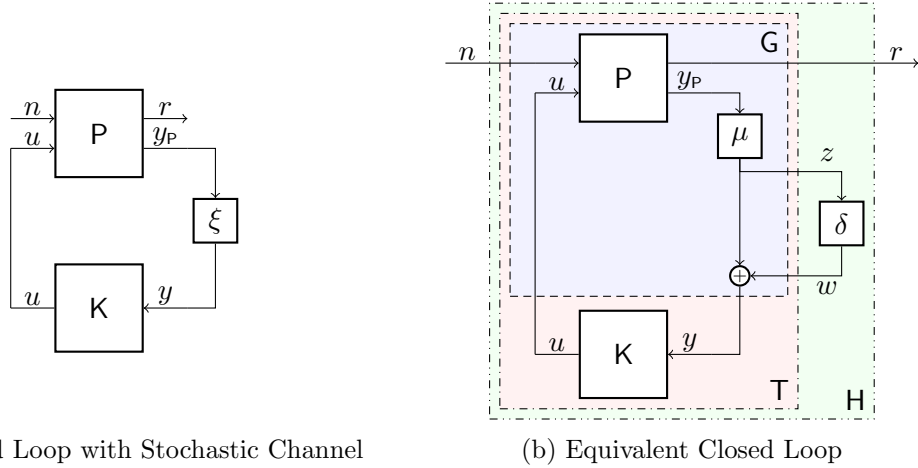


Figure 2.2: Equivalent Representations of a Stochastic Closed Loop

In this case the closed loop system will be stochastic and time-varying. Even if we restrict our attention to linear systems, the classical notion of stability we applied to the time-invariant case is no longer applicable. The closed loop matrices will be stochastic time-varying and the system state will be, even with zero input, a random process. Various notions of stability for stochastic systems have existed for some time [63]. Among the more common notions are mean stability, almost sure stability, and mean-square stability. In this work we will focus on mean-square (MS) stability, which we will see for the class of systems we consider is a strong notion of stability—e.g. the other two mentioned are implied by it but not vice versa.

We represent this kind of scenario as shown in Figure 2.2b by defining a zero-mean  $\delta(k)$  such that  $\xi(k) = \mu(1 + \delta(k))$ . This will allow us to isolate the stochastic uncertainty of the closed loop from an otherwise known, LTI mean system  $T$ . Importantly, this will generally

allow us to derive our theory with respect to the mean system and the fixed covariance of the uncertainty. Next we review the general framework we use as well as provide the key defining assumptions used throughout this work.

## 2.2 Fading Network Framework

We briefly review the general framework we use in this work which is based on the Fading Network Framework introduced in [23]. In Figure 2.3 we group the networked system into several main components: the generalized plant  $G : \begin{bmatrix} n \\ w \end{bmatrix} \mapsto \begin{bmatrix} r \\ z \end{bmatrix}$ , the controller  $K : y \mapsto u$ , and a stochastic operator  $\Delta : z \mapsto w$ . The generalized plant  $G$  is the interconnection of the plant(s) and the mean network interconnections where  $r$  is a regulated/performance output and  $n$  is an exogenous additive noise. The multiplicative noise  $w = \Delta z$  models stochastic channel fading at the physical layer or link dropouts or packet drops at the transport layer<sup>1</sup>. Using these components we define the mean closed loop  $T = F_\ell(G, K) : \begin{bmatrix} n \\ w \end{bmatrix} \mapsto \begin{bmatrix} r \\ z \end{bmatrix}$ , and the stochastic closed loop  $H = F_\ell(T, \Delta(k)) : n \mapsto r$ .

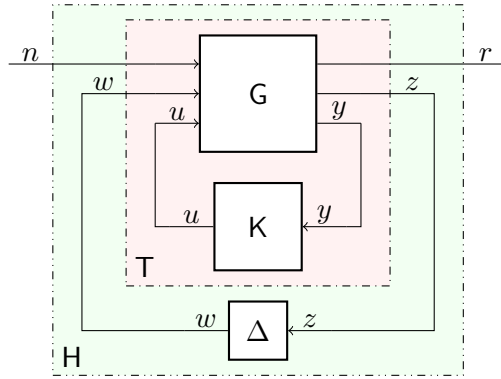


Figure 2.3: Fading network setup with exogenous noise  $n$  and regulated output  $r$

Notice that the framework we use isolates the stochastic uncertainty  $\Delta$  from an otherwise known system  $T$ , which allows us to study the overall time varying stochastic problem from an robust control perspective and gain further insights into the ways the stochastic uncertainty enters into and couples the dynamics.

<sup>1</sup>In this work we neglect quantization effects.

### 2.3 Standing Assumptions

We make several assumptions for any given  $H = F_\ell(\mathbb{T}, \Delta(k)) = F_\ell(F_\ell(\mathbf{G}, \mathbf{K}), \Delta)$  in this work:

**Assumption 1.**  $\mathbb{T}$  is finite-dimensional linear time-invariant (FDLTI).

**Assumption 2.**  $\Delta(k) = \text{diag}(\delta_i(k)I_{\eta_i})$  where  $\delta_i(k)$  are zero mean, finite variance, mutually independent, i.i.d. in time, random processes and  $\eta_i$  is the number of adjacent elements in  $\Delta(k)$  equal to  $\delta_i(k)$ .

**Assumption 3.** The exogenous input  $n(k)$  is a zero mean, finite variance, i.i.d. random process which is independent from  $\Delta(k) \forall k \geq 0$ .

Let  $\Sigma = E(\Delta(k)\mathbf{1}\mathbf{1}^T\Delta(k)^T) = \text{cov}(\Delta(k)\mathbf{1}) \in \mathbf{B}_+^n$  and let  $\mathbb{T}$  have a realization (2.2) where  $r(k) \in \mathbf{R}^{l_r}$ ,  $z(k) \in \mathbf{R}^{l_z}$ ,  $n(k) \in \mathbf{R}^{m_n}$ ,  $w(k) \in \mathbf{R}^{m_w}$  with  $l_z = m_w$ , and  $x(k) \in \mathbf{R}^n$ .

$$\mathbb{T} : \begin{bmatrix} x^+ \\ r \\ z \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_n & \bar{B}_w \\ \bar{C}_r & \bar{D}_{rn} & \bar{D}_{rw} \\ \bar{C}_z & \bar{D}_{zn} & \bar{D}_{zw} \end{bmatrix} \begin{bmatrix} x \\ n \\ w \end{bmatrix} \quad (2.2)$$

**Assumption 4.**  $x(0)$  is independent from both random processes  $\Delta(k)$  and  $n(k)$ .

**Assumption 5.**  $\bar{D}_{zw}$  in (2.2) is strictly block lower or upper triangular around the structure of  $\Sigma$ , i.e. it is nonzero only above or below the nonzero diagonal blocks of  $\Sigma$ .

We make Assumptions 1 and 2 since the notions of stability and performance we use as well as the underlying theory we will be extending in this work apply to finite-dimensional systems subject to perturbation by i.i.d. random processes. Assumption 4 coupled with the time independence in Assumptions 2 and 3 guarantees that  $x(k)$  will be independent of both  $n(k)$  and  $\Delta(k)$  at any given time  $k$ . Assumption 5 guarantees  $H = F_\ell(\mathbb{T}, \Delta)$  is well posed both stochastically and algebraically. Namely, it guarantees that  $\delta_i(k)$  and the elements of  $z(k)$  multiplied by  $\delta_i(k)$  in  $\Delta(k)z(k)$  are independent, and  $(I - \Delta(k)\bar{D}_{zw})^{-1}$  exists  $\forall k \geq 0$ .

Note that in Assumption 2 we allow for vector uncertainty, i.e. groups of channels to be affected by the same  $\delta_i$ . To aid the reader in understanding our notation we provide an example. Denoting  $\text{var}(\delta_i(k)) = \sigma_i^2$ ,  $\eta = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}$  corresponds to the following  $\Delta(k)$  and  $\Sigma$ .

$$\Delta(k) = \begin{bmatrix} \delta_1(k)I_3 & 0 & 0 \\ 0 & \delta_2(k) & 0 \\ 0 & 0 & \delta_3(k)I_2 \end{bmatrix} = \begin{bmatrix} \delta_1(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_1(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_1(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_2(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_3(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_3(k) \end{bmatrix} \in \mathbf{D}^6$$

$$\Sigma = \begin{bmatrix} \sigma_1^2 \mathbf{1}_{3 \times 3} & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \mathbf{1}_{2 \times 2} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_1^2 & \sigma_1^2 & 0 & 0 & 0 \\ \sigma_1^2 & \sigma_1^2 & \sigma_1^2 & 0 & 0 & 0 \\ \sigma_1^2 & \sigma_1^2 & \sigma_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_3^2 & \sigma_3^2 \\ 0 & 0 & 0 & 0 & \sigma_3^2 & \sigma_3^2 \end{bmatrix} \in \mathbf{B}_+^\eta$$

## 2.4 Mean-Square Stability and Performance

In this section we define the notions of stability and performance for stochastic systems that we use in this work. We begin by defining mean-square stability following [60].

**Definition 1.** Given  $\mathbf{H} = \mathcal{F}_\ell(\mathbb{T}, \Delta)$  where  $\mathbb{T}$  has a realization (2.2), let  $X(k) = \mathbb{E}(x(k)x^T(k))$  denote the state correlation matrix<sup>2</sup>. Then  $\mathbf{H}$  is mean-square (MS) stable if  $X(k)$  is well defined for all  $k \geq 0$  and  $n(k) = 0 \forall k \geq 0$  implies  $\lim_{k \rightarrow \infty} X(k) = 0$  for any  $X(0) \succeq 0$ .

For the class of systems we consider in this work, MS stability implies mean stability, i.e. that for zero exogenous input  $\mathbb{E}(x(k)) \rightarrow 0$  as  $k \rightarrow \infty$  for all initial conditions, and that  $x(k) \rightarrow 0$  as  $k \rightarrow \infty$  with probability one [60], i.e. *almost surely*. We point out that while MS

<sup>2</sup>We are borrowing this terminology from [60].

stability implies almost sure stability, the converse is not true. Finally we note that if  $\mathbf{H}$  is MS stable and  $n(k) \neq 0$ , e.g.  $\mathbb{E}(n(k)n(k)^T) = N$ , linearity implies that  $\lim_{k \rightarrow \infty} X(k) = \bar{X} \succeq 0$  for any  $X(0) \succeq 0$ . Next we define the notion of performance used in this work.

**Definition 2.** Given  $\mathbf{H} = \mathbb{F}_\ell(\mathbf{T}, \Delta)$ , let  $R(k) = \mathbb{E}(r(k)r^T(k))$  denote the performance correlation matrix. Then  $\mathbf{H}$  has mean-square (MS) performance  $\nu$  subject to  $n(k)$  if it is MS stable and  $\lim_{k \rightarrow \infty} \text{tr}(R(k)) = \nu^2$ .

Note our definition of MS performance includes MS stability. This is to avoid dealing with systems lacking internal MS stability having bounded performance due to hidden modes. Finally, we define a notation specifically for the MS performance of a system subject to exogenous noise with identity covariance.

**Definition 3.** Let  $\mathbb{E}(n(k)n(k)^T) = I$ , and define  $\|\mathbf{H}\|_{MSP}$  as

$$\|\mathbf{H}\|_{MSP} = \begin{cases} \nu & \text{if } \mathbf{H} \text{ has MS performance } \nu \\ \infty & \text{if } \mathbf{H} \text{ is not MS stable} \end{cases}$$

## CHAPTER 3. MEAN-SQUARE PERFORMANCE ANALYSIS

In this section we address MS performance analysis. First we show that the MS stability and MS performance properties of any given system  $H$  meeting the assumptions in Section 2.3 are equivalent to stability and performance properties of a linear time-invariant system of matrices. We use this to derive several important properties relating the the MS stability and performance of systems in this work. We then show that the MS stability and MS performance of any given system can be determined by a convex feasibility problem and a convex optimization problem, namely the feasibility of and optimization of a linear cost subject to LMIs. This result serves as the primary foundation for the synthesis theory later in this work.

### 3.1 Properties and Equivalent Conditions for MS Performance

We begin by providing a characterization of the state and performance correlation matrices for a given closed loop system which will be instrumental in deriving the subsequent analysis theory as well as the synthesis theory which follows.

**Lemma 1.** *Given  $H = F_\ell(\Upsilon, \Delta)$  where  $\Upsilon$  has a realization (2.2),  $\Sigma = \text{cov}(\Delta\mathbf{1})$ , subject to exogenous noise with  $\text{cov}(n(k)) = N$ , let  $X(k)$  and  $R(k)$  be as in Definitions 1 and 2. Then*

(a)  $\forall k \geq 0$  there exist  $Z(k) \triangleq E(z(k)z^T(k))$  and  $W(k) \triangleq E(w(k)w^T(k))$  such that

$$X(k+1) = \bar{A}X(k)\bar{A}^T + \bar{B}_n N \bar{B}_n^T + \bar{B}_w W(k) \bar{B}_w^T \quad (3.1a)$$

$$R(k) = \bar{C}_r X(k) \bar{C}_r^T + \bar{D}_{rn} N \bar{D}_{rn}^T + \bar{D}_{rw} W(k) \bar{D}_{rw}^T \quad (3.1b)$$

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \bar{D}_{zw} W(k) \bar{D}_{zw}^T \quad (3.1c)$$

$$W(k) = \Sigma \circ Z(k) \quad (3.1d)$$

(b)  $\mathbf{H}$  is MS stable if and only if  $\forall X(0) \succeq 0$ , (3.1a)-(3.1d) converge to a unique steady state solution  $\bar{X} \succeq 0$ ,  $\bar{R} \succeq 0$ ,  $\bar{Z} \succeq 0$ , and  $\bar{W} \succeq 0$  as  $k \rightarrow \infty$ .

(c)  $\mathbf{H}$  is MS stable iff  $\forall X(0) \succeq 0$ ,  $N = 0$  implies (3.1a)-(3.1d) converge to 0 as  $k \rightarrow \infty$

Proof in Appendix B.

In Lemma 1 we have that the MS stability and MS performance of a given linear time-varying system  $\mathbf{H}$  can be determined by investigating stability and performance properties of the matrix LTI system (3.1), which roughly speaking, is essentially a kind of feedback interconnection of a matrix version of  $\mathbf{T}$  and a constant  $\Sigma$ . Intuitively one might expect that this implies certain properties, such as input-output invariance with respect to a particular state space realization and stability/performance duality. We will show that we in fact do have these properties, but first we address the connection between Lemma 1 and  $\|\mathbf{H}\|_{MSP}$ .

Given Lemma 1 and recalling Definitions 2 and 3, we can verify by inspection that the MS performance of a system subject to exogenous noise with any finite covariance is equivalent to the MS performance norm of the system scaled by the square root of the noise covariance.

**Corollary 1.** *Let  $\mathbf{H} = \mathbf{F}_\ell(\mathbf{T}, \Delta)$  be MS stable. Then the MS performance of  $\mathbf{H}$  subject to exogenous noise with  $\text{cov}(n) = N$  is equal to  $\|\mathbf{H}N^{\frac{1}{2}}\|_{MSP}$ .*

Based on Corollary 1 we have no loss of generality by considering the case  $N = I$ , i.e. concentrating on  $\|\mathbf{H}\|_{MSP}$  for analysis purposes. To see why let  $\mathbf{H} = \mathbf{F}_\ell(\mathbf{T}, \Delta)$  where  $\mathbf{T}$  has a realization (2.2). Clearly the MS performance of  $\mathbf{H}$  subject to noise with  $\text{cov}(n) = N$  is equivalent to  $\|\dot{\mathbf{H}}\|_{MSP}$  where  $\dot{\mathbf{H}} = \mathbf{F}_\ell(\dot{\mathbf{T}}, \Delta)$  and

$$\dot{\mathbf{T}} : \begin{bmatrix} \dot{x}^+ \\ \dot{r} \\ \dot{z} \end{bmatrix} : \left[ \begin{array}{c|cc} \bar{A} & \dot{B}_n & \bar{B}_w \\ \hline \bar{C}_r & \dot{D}_{rn} & \bar{D}_{rw} \\ \bar{C}_z & \dot{D}_{zn} & \bar{D}_{zw} \end{array} \right] \begin{bmatrix} \dot{x} \\ \dot{n} \\ \dot{w} \end{bmatrix} \quad (3.2)$$

with  $\dot{B}_n = \bar{B}_n N^{\frac{1}{2}}$ ,  $\dot{D}_{rn} = \bar{D}_{rn} N^{\frac{1}{2}}$ , and  $\dot{D}_{zn} = \bar{D}_{zn} N^{\frac{1}{2}}$ .

Now as alluded to earlier we derive two additional, relatively intuitive but crucial corollaries: first we show that the parts of the steady state solution in Lemma 1(b) corresponding to input/output signals to  $\mathsf{T}$  do not change based on the state space realization of  $\mathsf{T}$ , and second we show that the MS stability and MS performance of a given system are equivalent to that of the dual system<sup>1</sup>.

**Corollary 2.** *Given a MS stable  $\mathsf{H} = \mathsf{F}_\ell(\mathsf{T}, \Delta)$  with  $\Sigma = \text{cov}(\Delta\mathbf{1})$ , the quantities  $\bar{R}$ ,  $\bar{Z}$ , and  $\bar{W}$  in Lemma 1(b) are invariant with respect to the particular state space realization of  $\mathsf{T}$ .*

Proof in Appendix B.

**Corollary 3.** *Given  $\mathsf{H} = \mathsf{F}_\ell(\mathsf{T}, \Delta)$  and  $\Sigma = \text{cov}(\Delta\mathbf{1})$ , the dual system  $\mathsf{H}^T = \mathsf{F}_\ell(\mathsf{T}^T, \Delta)$  is such that  $\|\mathsf{H}\|_{\text{MSP}} = \|\mathsf{H}^T\|_{\text{MSP}}$ .*

Proof in Appendix B.

Corollary 2 will be crucial later on when we derive a separation principle for the synthesis problem in Section 6. We will take advantage of the results in Corollary 3 extensively throughout this work. We will often simply say something follows by duality.

Now we point out that although Lemma 1 provides a theoretical condition for MS stability and for obtaining the MS performance for a MS stable system, and is quite useful in the derivation of important properties, it is not practical in the sense that one would have to check an infinite number of initial conditions to ensure MS stability. To address this, we derive an linear matrix inequality (LMI) based characterization of MS stability and performance.

**Definition 4.** *Given  $\mathsf{H} = \mathsf{F}_\ell(\mathsf{T}, \Delta)$  where  $\mathsf{T}$  has a realization (2.2) and  $\Sigma = \text{cov}(\Delta\mathbf{1}) \in \mathbf{B}_+^\eta$ , let  $\mathbf{L}$  be the feasible set of all  $X \in \mathbf{S}_{++}^n$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$  and  $W \in \mathbf{B}_{++}^\eta$  satisfying the LMIs*

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_wW\bar{B}_w^T \quad (3.3a)$$

$$R \succ \bar{C}_rX\bar{C}_r^T + \bar{D}_{rn}\bar{D}_{rn}^T + \bar{D}_{rw}W\bar{D}_{rw}^T \quad (3.3b)$$

$$Z \succ \bar{C}_zX\bar{C}_z^T + \bar{D}_{zn}\bar{D}_{zn}^T + \bar{D}_{zw}W\bar{D}_{zw}^T \quad (3.3c)$$

$$W \succ \Sigma \circ Z \quad (3.3d)$$

<sup>1</sup>Recall (1.1) and (1.2).



Similarly let  $\mathbf{L}'$  be the feasible set of all  $\tilde{X} \in \mathbf{S}_{+++}^n$ ,  $\tilde{R} \in \mathbf{S}_{+++}^{m_n}$ ,  $\tilde{Z} \in \mathbf{S}_{+++}^{l_z}$  and  $\tilde{W} \in \mathbf{B}_{++}^\eta$  satisfying the dual LMIs

$$\tilde{X} \succ \bar{A}^T \tilde{X} \bar{A} + \bar{C}_r^T \tilde{X} \bar{C}_r + \bar{C}_z^T \tilde{W} \bar{C}_z \quad (3.4a)$$

$$\tilde{R} \succ \bar{B}_n^T \tilde{X} \bar{B}_n + \bar{D}_{rn}^T \tilde{X} \bar{D}_{rn} + \bar{D}_{zn}^T \tilde{W} \bar{D}_{zn} \quad (3.4b)$$

$$\tilde{Z} \succ \bar{B}_w^T \tilde{X} \bar{B}_w + \bar{D}_{rw}^T \tilde{X} \bar{D}_{rw} + \bar{D}_{zw}^T \tilde{W} \bar{D}_{zw} \quad (3.4c)$$

$$\tilde{W} \succ \Sigma \circ \tilde{Z} \quad (3.4d)$$

**Theorem 1.** Given  $\mathbf{H} = F_\ell(\mathbb{T}, \Delta)$  where  $\mathbb{T}$  has a realization (2.2) and  $\Sigma = \text{cov}(\Delta \mathbf{1}) \in \mathbf{B}_+^\eta$ ,

$$(a) \quad \|\mathbf{H}\|_{MSP}^2 < \nu^2 \Leftrightarrow \exists (X, R, Z, W) \in \mathbf{L} \text{ such that } \nu^2 = \text{tr}(R)$$

$$(b) \quad \|\mathbf{H}\|_{MSP}^2 < \nu^2 \Leftrightarrow \exists (\tilde{X}, \tilde{R}, \tilde{Z}, \tilde{W}) \in \mathbf{L}' \text{ such that } \nu^2 = \text{tr}(\tilde{R}).$$

$$(c) \quad \|\mathbf{H}\|_{MSP}^2 = \inf_{\mathbf{L}} \text{tr}(R) = \inf_{\mathbf{L}'} \text{tr}(\tilde{R})$$

Proof in Appendix B.

The following corollary can be verified by inspection of the LMIs in Theorem 1, and will be instrumental later on when we are deriving results related to the Youla parameterization of all MS stable closed loops for synthesis problems.

**Corollary 4.**  $\mathbf{H} = F_\ell(\mathbb{T}, \Delta)$  is MS stable only if  $\mathbb{T}$  is stable.

Theorem 1 generalizes the MS stability analysis theory in [23] to include MS performance and cases of vector uncertainty, i.e. groups of signals affected by the same random variable. Recently [37] derived a quasi-convex spectral radius method for calculating MS performance. The advantages of our LMI formulation over [37] are that it provides a convex rather than quasi-convex method of calculating MS performance since the matrix inequalities in Theorem 1 are linear in  $\nu^2$ , and it can handle vector fading. Additionally, the inequalities are also linear for any fixed  $\Sigma$ , which means various channel analysis and resource allocation problems are quasi-convex. Finally, and perhaps most importantly, the LMI formulation in Theorem 1 can be adapted to a number of synthesis problems via simple change of coordinates. This will be central to the development in Section 5.

### 3.2 Simplified Equivalent Conditions for MS Stability

Before moving on, we briefly state a simplified version of Theorem 1 which results when only MS stability is a concern.

**Corollary 5.** *Given  $H = F_\ell(\mathsf{T}, \Delta)$  where  $\mathsf{T}$  has a realization (2.2) and  $\Sigma = \text{cov}(\Delta \mathbf{1}) \in \mathbf{B}_+^\eta$ ,  $H$  is MS stable if and only if*

(a) *there exist  $X \in \mathbf{S}_{++}^n$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ , and  $W \in \mathbf{B}_{++}^\eta$  such that*

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_wW\bar{B}_w^T \quad (3.5a)$$

$$Z \succ \bar{C}_zX\bar{C}_z^T + \bar{D}_{zw}W\bar{D}_{zw}^T \quad (3.5b)$$

$$W \succ \Sigma \circ Z \quad (3.5c)$$

(b) *there exist  $\tilde{X} \in \mathbf{S}_{++}^n$ ,  $\tilde{Z} \in \mathbf{S}_{++}^{l_z}$  and  $\tilde{W} \in \mathbf{B}_{++}^\eta$  such that*

$$\tilde{X} \succ \bar{A}^T\tilde{X}\bar{A} + \bar{C}_z^T\tilde{W}\bar{C}_z \quad (3.6a)$$

$$\tilde{Z} \succ \bar{B}_w^T\tilde{X}\bar{B}_w + \bar{D}_{zw}^T\tilde{W}\bar{D}_{zw} \quad (3.6b)$$

$$\tilde{W} \succ \Sigma \circ \tilde{Z} \quad (3.6c)$$

The LMIs in Corollary 5 functionally recover those in [23, 27] for the special case  $\Sigma \in \mathbf{D}_+^q$ .

## CHAPTER 4. NETWORKED CONTROL SYNTHESIS

In this chapter we provide the formulation for the main control synthesis problem of this work: a networked control system where multiple sensor and actuator signals are transmitted through stochastic links, and both the signals and plant are subject to exogenous additive noise.

Figure 4.1 shows the type closed loop system we consider. The controller  $\mathcal{K}$  interacts with the plant  $\mathcal{P}$  via unreliable network connections  $\Xi_a$  and  $\Xi_s$ , and has access to channel state information  $\alpha_a$  and  $\alpha_s$  where  $z^{-1}$  denotes a one step delay. This is the same remote control setup previously studied from a MS stability perspective in [28] where now we will consider the effects of stochastic exogenous noise  $n$  on the closed loop MS performance as well as less conservative assumptions for the network connections.

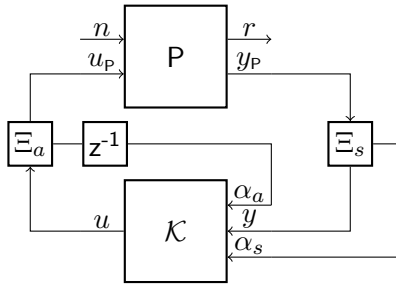


Figure 4.1: Networked Control Synthesis Setup

### 4.1 Problem Formulation

We assume that the feedback interconnection of the plant and controller is well posed, and that the plant  $\mathcal{P}$  is FDLTI system with a realization (4.1).

$$\mathcal{P} : \begin{bmatrix} x_p^+ \\ r \\ y_p \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_n & B_u \\ \hline C_r & D_{rn} & D_r \\ C_y & D_n & 0 \end{array} \right] \begin{bmatrix} x_p \\ n \\ u_p \end{bmatrix} \quad (4.1)$$

We model the unreliable network connections as multiplicative uncertainties with known nonzero mean and finite variance:  $\Xi_s(k) = \text{diag}(\xi_{s_i}(k)I_{\eta_{s_i}})$  and  $\Xi_a(k) = \text{diag}(\xi_{a_j}(k)I_{\eta_{a_j}})$  with  $\{\xi_{s_i}(k), \xi_{a_j}(k)\}$  mutually independent, i.i.d. processes, and  $\eta_{s_i}$  and  $\eta_{a_j}$  the sizes of vectors affected by  $\xi_{s_i}$  and  $\xi_{a_j}$ .

**Remark 1.** *We point out that it is more typical to consider the special case of packet drops modeled as Bernoulli processes. While this is the most common and practical scenario from a NCS with receiver acknowledgment perspective, we derive our theory in a more general setting. This allows us to show it is not restricted to Bernoulli—or generally to discrete—processes. This generality may also prove useful for scenarios beyond the NCS setting considered in this work.*

We restrict all processing to be done locally at the controller  $\mathcal{K}$  which has access the channel state information provided by  $\alpha_a(k)$  and  $\alpha_s(k)$ . We assume that while the sensor receiver can instantaneously report the channel state, i.e.  $\alpha_s(k) = \xi_s(k)$ , the actuator receiver is subject to one step delay, i.e.  $\alpha_a(k) = \xi_a(k-1)$ . This delay accounts for the physical propagation delays which prevent the controller from knowing the result of the transmission before it has happened.

#### 4.1.1 Admissible Class of Controllers

Motivated by [28, 30], we restrict our controllers to be those with control action equivalent to (4.2).

$$\mathcal{K} : \begin{bmatrix} x_{\mathcal{K}}^+ \\ u \end{bmatrix} = \left[ \begin{array}{c|c} \frac{A_{\mathcal{K}} + B_{\mathcal{K}_1}\Xi_s C_{\mathcal{K}_1} + B_{\mathcal{K}_2}\Xi_a C_{\mathcal{K}_2} + B_{\mathcal{K}_2}\Xi_s D_{\mathcal{K}}\Xi_s C_{\mathcal{K}_1}}{C_{\mathcal{K}_2} + D_{\mathcal{K}}\Xi_s C_{\mathcal{K}_1}} & \frac{B_{\mathcal{K}_1} + B_{\mathcal{K}_2}\Xi_a D_{\mathcal{K}}}{D_{\mathcal{K}}} \end{array} \right] \begin{bmatrix} x_{\mathcal{K}} \\ y \end{bmatrix} \quad (4.2)$$

It is clear that these controllers will scale in an essentially linear fashion with the number of random channel states, i.e. the dimensions of  $\Xi_a$  and  $\Xi_s$ , and that they do not have any sample path dependence and therefore the gains can be calculated offline. As previously pointed out in Section 1.2, these are important contrasts with related approaches in e.g. Markov Jump Linear Systems literature which allow for the controller matrices to switch based on every possible combination of channel states, as well as with any approaches using Kalman filtering which have sample path dependence for the problems we consider.

We note that the controller state  $x_{\kappa}(k)$  and the command output  $u(k)$  at time  $k$  do not depend on knowledge of  $\Xi_a(k)$ , which we assume we do not have, but rather only on  $\Xi_a(k-1)$ , which we do. This means that controllers equivalent to (4.2) are implementable given our information constraints.

It can be easily verified that the LTV controller in (4.2) is equivalent to the LTI controller with time-varying feedback loops defined in (4.3) where  $u_2(k) \triangleq u(k)$ .

$$\mathbf{K}: \begin{bmatrix} x_{\kappa}^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|cc} A_{\kappa} & B_{\kappa_1} & B_{\kappa_2} \\ \hline C_{\kappa_1} & 0 & 0 \\ C_{\kappa_2} & D_{\kappa} & 0 \end{array} \right] \begin{bmatrix} x_{\kappa} \\ y_1 \\ y_2 \end{bmatrix} \quad (4.3a)$$

$$y_1(k) = y(k) + \Xi_s(k)u_1(k) \quad (4.3b)$$

$$y_2(k) = \Xi_a(k)u_2(k) \quad (4.3c)$$

Moreover, using standard LTI systems manipulations, the controller structure defined in (4.3) is equivalent to that in (4.4) by factoring a delay out of the second input.

$$\hat{\mathbf{K}}: \begin{bmatrix} x_{\kappa}^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|cc} A_{\kappa} & B_{\kappa_1} & A_{\kappa}B_{\kappa_2} \\ \hline C_{\kappa_1} & 0 & C_{\kappa_1}B_{\kappa_2} \\ C_{\kappa_2} & D_{\kappa} & C_{\kappa_2}B_{\kappa_2} \end{array} \right] \begin{bmatrix} x_{\kappa} \\ y_1 \\ \hat{y}_2 \end{bmatrix} \quad (4.4a)$$

$$y_1(k) = y(k) + \Xi_s(k)u_1(k) \quad (4.4b)$$

$$\hat{y}_2 = y_2^- = y_2(k-1) = \Xi_a(k-1)u_2(k-1) \quad (4.4c)$$

This version provides an implementable but equivalent control action to that in (4.2): requiring only that the the previous command  $u_2(k-1)$  be stored in memory and used with the information provided by  $\alpha_a(k)$  to produce  $\hat{y}_2(k)$ .



with actuator and sensor uncertainty represented by  $\Delta(k) = \text{diag}(\Delta_s(k), \Delta_a(k)) : z(k) \mapsto w(k)$  with  $\Sigma = \text{diag}(\Sigma_s, \Sigma_a) = \text{cov}(\Delta(k)\mathbf{1}) \in \mathbf{B}_+^n$ . Let  $\mathbf{K}$  be the set of all FDLTI controllers with a realization (4.6) making  $\mathbf{H} = \mathbb{F}_\ell(\mathbb{F}_\ell(\mathbf{G}, \mathbf{K}), \Delta)$  MS stable, and  $\nu^* \triangleq \inf_{\mathbf{K} \in \mathbf{K}} \|\mathbf{H}\|_{MSP}$ .

$$\mathbf{K}: \begin{bmatrix} x_k^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|cc} A_k & B_{k_1} & B_{k_2} \\ \hline C_{k_1} & 0 & 0 \\ C_{k_2} & D_k & 0 \end{array} \right] \begin{bmatrix} x_k \\ y_1 \\ y_2 \end{bmatrix} \quad (4.6)$$

We seek the controller  $\mathbf{K}$  which gives the optimal MS performance closed loop in Figure 4.2. Controller synthesis is expected to be non-convex in the general setting of Figure 2.3. This is due to the analogy of the setup with robust control synthesis [23]. The difficulties in finding a convex synthesis for Problem 1 are confirmed when trying adapt the LMIs of Theorem 1 for synthesis based on the standard change of coordinates [64]. While this method makes the synthesis for many problems convex, including standard  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$ , it does not do so here. Due to additional problem variables resulting from the channel uncertainty, not all nonlinearities can be eliminated by this change of variables or any modification of it we are aware of.

We will show that under mild conditions our optimal MS performance synthesis can be solved convexly via separation analogous to the classical  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  [65]. We will see in what follows that this is highly nontrivial, and does not follow simply from combining existing approaches, e.g. the static gain observer results in [9, 10, 12] with state feedback, or directly using classical  $\mathcal{H}_2$  techniques.

The remainder of this work is primarily devoted to building the necessary machinery to attack, then finally solve this problem. Preliminary results are given in [27, 28, 30, 33, 35, 36].

## CHAPTER 5. NETWORKED SPECIAL PROBLEMS

In this chapter we investigate a sequence of problems analogous to the classical special problems: Full Information, Disturbance Feedforward, Full Control, and Output Estimation [61]. In the LTI setting, these problems form the basis for the Youla parameterization of all stabilizing controllers and closed loops for the classical output feedback problem as well as the derivation of the  $\mathcal{H}_2$  separation principle [61,66].

Under conditions which are automatically satisfied in the standard output feedback scenario, the Full Information and Disturbance Feedforward problems are equivalent in the sense that they have equal optimal  $\mathcal{H}_2$  performance and the same optimal performance closed loop. Similarly, the Full Control and Output Estimation problems are equivalent, and are the dual of the Full Information and Disturbance Feedforward problems respectively.

We will see that not only do the networked special problems we consider share equivalence and duality properties analogous to the classical versions, but they will be just as essential in the derivation of the networked output feedback synthesis theory in Chapter 6.

We begin by defining notions of stabilizability and detectability for given system matrices.

**Definition 5.** *The quadruple  $(A, B_w, B_u, \Sigma)$  is MS stabilizable if there exists an  $F$  for which  $A_F : \mathbf{S}_+^n \mapsto \mathbf{S}_+^n$  given by  $A_F(X) = (A + B_u F)X(A + B_u F)^T + B_w(\Sigma \circ (F X F^T))B_w^T$  has  $\rho(A_F) < 1$ .*

**Definition 6.** *The quadruple  $(A, C_z, C_y, \Sigma)$  is MS detectable if there exists an  $L$  for which  $A_L : \mathbf{S}_+^n \mapsto \mathbf{S}_+^n$  given by  $A_L(X) = (A + LC_y)^T X(A + LC_y) + C_z^T(\Sigma \circ (L^T X L))C_z$  has  $\rho(A_L) < 1$ .*

The classical analog to Definition 5 is that the pair  $(A, B_u)$  is stabilizable if there exists an  $F$  such that  $A + B_u F$  is Schur. Requiring  $A + B_u F$  to be Schur is equivalent to requiring  $\rho(A + B_u F) < 1$  since the spectral radius of a matrix is trivially the maximum magnitude eigenvalue. Therefore quite sensibly, Definition 5 is equivalent to the classical definition of



$(A, B_u)$  stabilizable when  $\Sigma = 0$ . This implies for example that  $(A, B_w, B_u, \Sigma)$  is MS stabilizable only if  $(A, B_u)$  is stabilizable. Of course we have the same connections between Definition 6 and the classical notion of a pair  $(A, C_y)$  being detectable.

As we proceed through this chapter we will see that  $(A, B_w, B_u, \Sigma)$  MS stabilizable is necessary and sufficient condition for the existence of MS stabilizing controllers for the networked Full Information (FI) and Disturbance Feedforward (DF) problems to follow, and similarly that  $(A, C_z, C_y, \Sigma)$  MS detectable is equivalent to the existence of MS stabilizing controllers for the networked Full Control (FC) and Output Estimation (OE) problems. Additionally, we will show in Chapter 6 that combined MS stabilizability and MS detectability will be equivalent to the existence of MS stabilizing controllers.

## 5.1 Networked Full Information and Disturbance Feedforward

In this section we investigate networked extensions of the classical FI and DF problems. The networked FI and DF scenarios are depicted in Figure 5.1a and 5.1b respectively. Notice that in each case we consider the actuator link subject to stochastic uncertainty. We formally define these problems next.

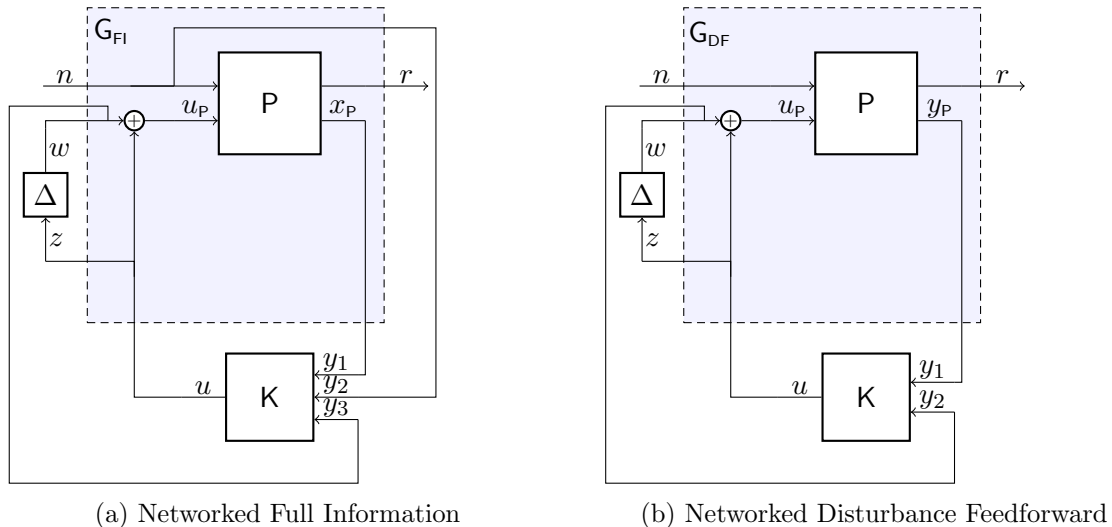


Figure 5.1: Networked FI and DF Synthesis Setups

**Problem 2** (Networked Full Information). *Let the networked FI synthesis problem have generalized plant given by*

$$\mathbf{G}_{\text{FI}}: \begin{bmatrix} x_G^+ \\ r \\ z \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} A & B_n & B_w & B_u \\ \hline C_r & D_{rn} & D_{rw} & D_r \\ 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w \\ u \end{bmatrix} \quad (5.1)$$

with actuator uncertainty represented by  $\Delta(k) : z(k) \mapsto w(k)$  having  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$ . Let  $\mathbf{K}_{\text{FI}}$  be the set of all FDLTI controllers making  $\mathbf{H}_{\text{FI}} = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{G}_{\text{FI}}, \mathbf{K}_{\text{FI}}), \Delta)$  MS stable which are strictly proper from  $y_3$  to  $u$ , and let  $\nu_{\text{FI}}^* \triangleq \inf_{\mathbf{K}_{\text{FI}} \in \mathbf{K}_{\text{FI}}} \|\mathbf{H}_{\text{FI}}\|_{\text{MSP}}$ .

**Problem 3** (Networked Disturbance Feedforward). *Let the networked DF synthesis problem have generalized plant given by*

$$\mathbf{G}_{\text{DF}}: \begin{bmatrix} x_G^+ \\ r \\ z \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B_n & B_w & B_u \\ \hline C_r & D_{rn} & D_{rw} & D_r \\ 0 & 0 & 0 & I \\ C_y & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w \\ u \end{bmatrix} \quad (5.2)$$

with actuator uncertainty represented by  $\Delta(k) : z(k) \mapsto w(k)$  having  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$ . Let  $\mathbf{K}_{\text{DF}}$  be the set of all FDLTI controllers making  $\mathbf{H}_{\text{DF}} = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{G}_{\text{DF}}, \mathbf{K}_{\text{DF}}), \Delta)$  MS stable which are strictly proper from  $y_2$  to  $u$ , and let  $\nu_{\text{DF}}^* = \inf_{\mathbf{K}_{\text{DF}} \in \mathbf{K}_{\text{DF}}} \|\mathbf{H}_{\text{DF}}\|_{\text{MSP}}$ .

Next we show that the existence of a MS stabilizing controller for the networked Full Information synthesis in Problem 2 is equivalent to the feasibility of a set of linear matrix inequalities (LMIs), and moreover that we can obtain a static controller with MS performance arbitrarily close to the optimal from feasible variables.

**Definition 7.** Given  $\mathbf{G}_{\text{FI}}$  in (5.1) or  $\mathbf{G}_{\text{DF}}$  in (5.2) with  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$  let  $\mathbf{M}_{\text{F}}$  be the feasible set of all  $X \in \mathbf{S}_{++}^n$ ,  $Y \in \mathbf{R}^{m_u \times n}$ ,  $J \in \mathbf{R}^{m_u \times m_n}$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  satisfying the LMIs (5.3a)-(5.3d).

$$\begin{bmatrix} X & (AX+B_uY) & (B_n+B_uJ) & B_wW \\ (AX+B_uY)^T & X & 0 & 0 \\ (B_n+B_uJ)^T & 0 & I & 0 \\ (B_wW)^T & 0 & 0 & W \end{bmatrix} \succ 0 \quad (5.3a)$$

$$\begin{bmatrix} R & (CX+D_rY) & (D_1+D_rJ) & D_2W \\ (CX+D_rY)^T & X & 0 & 0 \\ (D_{rn}+D_rJ)^T & 0 & I & 0 \\ (D_{rw}W)^T & 0 & 0 & W \end{bmatrix} \succ 0 \quad (5.3b)$$

$$\begin{bmatrix} Z & Y & J \\ Y^T & X & 0 \\ J^T & 0 & I \end{bmatrix} \succ 0 \quad (5.3c)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.3d)$$

**Lemma 2.** Given Problem 2, let  $\bar{\mathbf{K}}_{\text{FI}} \subseteq \mathbf{K}_{\text{FI}}$  be the admissible static controllers, i.e. all  $\bar{\mathbf{K}}_{\text{FI}} \in \mathbf{K}_{\text{FI}}$  with a realization (5.4), and let  $\mathbf{M}_{\text{F}}$  be as given in Definition 7. Then

$$\bar{\mathbf{K}}_{\text{FI}} : u = \begin{bmatrix} F & F_0 & 0 \end{bmatrix} y \quad (5.4)$$

$$(a) \quad \mathbf{K}_{\text{FI}} \neq \emptyset \Leftrightarrow \bar{\mathbf{K}}_{\text{FI}} \neq \emptyset \Leftrightarrow \mathbf{M}_{\text{F}} \neq \emptyset$$

$$(b) \quad (X, Y, J, R, Z, W) \in \mathbf{M}_{\text{F}} \Rightarrow \bar{\mathbf{K}}_{\text{FI}} = [YX^{-1} \ J \ 0] \in \bar{\mathbf{K}}_{\text{FI}} \text{ and } \|F_\ell(F_\ell(\mathbf{G}_{\text{FI}}, \bar{\mathbf{K}}_{\text{FI}}), \Delta)\|_{\text{MSP}}^2 < \text{tr}(R)$$

$$(c) \quad (\nu_{FI}^*)^2 = \inf_{\mathbf{M}_{\text{F}}} \text{tr}(R)$$

Proof in Appendix C.

Now we can show that the existence of a MS stabilizing controller for the networked Full Information problem is equivalent to  $(A, B_w, B_u, \Sigma)$  MS stabilizable, which moreover is equivalent to the feasibility of a simplified set of LMIs.

**Corollary 6.**  $\mathbf{K}_{\text{FI}} \neq \emptyset$  if and only if  $(A, B_w, B_u, \Sigma)$  is MS stabilizable, and  $(A, B_w, B_u, \Sigma)$  is MS stabilizable if and only if there exist  $X \in \mathbf{S}_{++}^n$ ,  $Y \in \mathbf{R}^{m_u \times n}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  such that

$$\begin{bmatrix} X & (AX+B_uY) & B_wW \\ (AX+B_uY)^T & X & 0 \\ (B_wW)^T & 0 & W \end{bmatrix} \succ 0 \quad (5.5a)$$

$$\begin{bmatrix} Z & Y \\ Y^T & X \end{bmatrix} \succ 0 \quad (5.5b)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.5c)$$

The static controller  $\bar{\mathbf{K}}_{\text{FI}} = [YX^{-1} \ 0 \ 0] \in \bar{\mathbf{K}}_{\text{FI}}$  for any feasible  $X$  and  $Y$ .

Proof in Appendix C.

Corollary 6 implies that the networked FI problem has a MS stabilizing solution if and only if it can be MS stabilized via state feedback, which can be determined by checking the feasibility of a simple set of LMIs: (5.5a)-(5.5c). We will see that the dual problem of networked Full Control will have analogous properties in Section 5.2.

The significance of this may not seem like much at this point, but we will see as we move on that due to the equivalence of the networked FI and FC problems to the networked DF and OE problems, and later the equivalence of the networked output feedback problem in Chapter 6 to a pair of special problems, that we can investigate the existence and limitations of MS stabilizing controllers for the any synthesis problem in this work using compact LMIs.

Next we investigate the limit points of the control gains  $F = YX^{-1}$  and  $F_0 = J$  in Lemma 2 as  $\text{tr}(R) \rightarrow (\nu_{F_I}^*)^2$ . We provide sufficient conditions for the limit points to be well defined and MS stabilizing. We will see that under the given conditions, the optimal control gains  $F^*$  and  $F_0^*$  as well as the optimal MS performance cost  $\nu_{F_I}^*$  can be obtained from the unique positive semidefinite solution to Riccati-like equations. We note that these equations, while having many analogous properties to the standard Riccati equations, are not like their classical counterparts solvable in terms of a matrix pencil [67].

**Lemma 3.** Given Problem 2 with  $(A, B_w, B_u, \Sigma)$  MS stabilizable,  $D_r^T D_r + \Sigma \circ D_{r_w}^T D_{r_w} \succ 0$ ,  $D_r^T C_r = 0$ , and  $(C_r, A)$  detectable,

(a) there exists a unique  $X^* \succeq 0$  and  $R^* \succeq 0$ :

$$X^* = A^T X^* A + C_r^T C_r - (A^T X^* B_u) S^{-1} (B_u^T X^* A) \quad (5.6a)$$

$$R^* = B_n^T X^* B_n + D_{rn}^T D_{rn} - (B_n^T X^* B_u + D_{rn}^T D_r) S^{-1} (B_u^T X^* B_n + D_r^T D_{rn}) \quad (5.6b)$$

$$S = B_u^T X^* B_u + D_r^T D_r + \Sigma \circ (B_w^T X^* B_w + D_{rw}^T D_{rw}) \quad (5.6c)$$

and controller gains

$$F^* = -S^{-1} (B_u^T X^* A) \quad (5.7a)$$

$$F_0^* = -S^{-1} (B_u^T X^* B_n + D_r^T D_{rn}) \quad (5.7b)$$

such that  $\|H_{FI}\|_{MSP} = \nu_{FI}^* = \text{tr}(R^*)^{\frac{1}{2}}$ .

(b) for any  $X(0) \succeq 0$ ,  $X^* = \lim_{k \rightarrow \infty} X(k)$  where

$$S(k) = B_u^T X(k) B_u + D_r^T D_r + \Sigma \circ (B_w^T X(k) B_w + D_{rw}^T D_{rw}) \quad (5.8a)$$

$$X(k+1) = A^T X(k) A + C_r^T C_r - (A^T X(k) B_u) S(k)^{-1} (B_u^T X(k) A) \quad (5.8b)$$

Proof in Appendix C.

Based on Lemma 3 we can conclude that optimal MS performance closed loop for the networked FI problem is  $H_{FI}^* = F_\ell(\mathbb{T}_{FI}^*, \Delta) = F_\ell(F_\ell \mathbf{G}_{FI}, \mathbf{K}_{FI}^*, \Delta)$  where

$$\mathbb{T}_{FI}^* : \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A + B_u F^* & B_n + B_u F_0^* & B_w \\ \hline C_r + D_r F^* & D_{rn} + D_r F_0^* & D_{rw} \\ F^* & F_0^* & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (5.9)$$

$$\mathbf{K}_{FI}^* : u = \begin{bmatrix} F^* & F_0^* & 0 \end{bmatrix} y \quad (5.10)$$

Besides the optimal closed loop, we can also recover any suboptimal MS performance closed loop via Youla-parameterization. This follows directly from standard results on Youla-parameterization of all stable closed loops, e.g. Chapter 12 of [61], combined with Corollary 4 in Chapter 3. That is, since any MS stable stochastic closed loop must have a stable mean closed loop, any mean closed loop which will be MS stable in feedback with  $\Delta$  is within a subset of all stable mean closed loops.

**Proposition 1.** *If  $F$  is such that  $A + B_u F$  is Schur, then*

(a) *any stable, FDLTI  $T_{F_I} = F_\ell(G_{F_I}, K_{F_I})$  can be realized as  $T_{F_I} = F_\ell(U_{F_I}, Q)$  where  $U_{F_I}$  has the realization*

$$U_{F_I} : \begin{bmatrix} \hat{x}^+ \\ r \\ z \\ n \\ w \end{bmatrix} = \left[ \begin{array}{c|ccc} A+B_u F & B_n & B_w & B_u \\ \hline C_r+D_r F & D_{rn} & \bar{D}_{rw} & D_r \\ F & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \\ q \end{bmatrix} \quad (5.11)$$

(b) *any MS stable  $H_{F_I} = F_\ell(T_{F_I}, \Delta)$  can be realized as  $H_{F_I} = F_\ell(F_\ell(U_{F_I}, Q), \Delta)$*

for some stable, FDLTI system  $Q$ .

It is easily verified that for any  $T_{F_I} = F_\ell(G_{F_I}, K_{F_I})$  and  $Q$  where  $T_{F_I} = F_\ell(U_{F_I}, Q)$ , if  $K_{F_I} \in \mathbf{K}_{F_I}$  then  $Q$  must have the structure (5.12).

$$Q : \begin{bmatrix} x_Q^+ \\ q \end{bmatrix} = \left[ \begin{array}{c|cc} A_Q & B_{Q_1} & B_{Q_2} \\ \hline C_Q & D_{Q_1} & 0 \end{array} \right] \begin{bmatrix} x_Q \\ n \\ w \end{bmatrix} \quad (5.12)$$

with  $A_Q$  Schur.

Finally before addressing the networked DF problem, we point out that the optimal control in Lemma 3 is in fact optimal for a set of networked FI closed loops: namely those which are equivalently affected by exogenous noise having any positive definite covariance.

**Proposition 2.** *Given a networked FI synthesis in Problem 3 where  $G_{F_I}$  in (5.1) meets the conditions of Lemmas 3, let  $F^*$ ,  $F_0^*$  be as given in (5.7), and  $K_{F_I^*}$  be as given in (5.10). Then*

$$\|F_\ell(F_\ell(G_{F_I}, K_{F_I}^*), \Delta) N^{\frac{1}{2}}\|_{MSP} \leq \|F_\ell(F_\ell(G_{F_I}, K_{F_I}), \Delta) N^{\frac{1}{2}}\|_{MSP} \quad (5.13)$$

for any  $N \succ 0$  and any  $K_{F_I} \in \mathbf{K}_{F_I}$ .

Proof in Appendix C.

Now we move on to the networked DF problem. We will see that analogous to the classical case, under a standard assumption it is equivalent to networked FI.

**Lemma 4.** Given Problem 3 with  $A-B_nC_y$  Schur, let  $\mathbf{M}_F$  be as given in Definition 7. Then

(a)  $\mathbf{K}_{DF} \neq \emptyset \Leftrightarrow \mathbf{M}_F \neq \emptyset$

(b) if  $(X, Y, J, R, Z, W) \in \mathbf{M}_F$ ,  $F = YX^{-1}$ ,  $F_0 = J$ , and

$$\bar{\mathbf{K}}_{DF} : \begin{bmatrix} x_K^+ \\ u \end{bmatrix} = \left[ \begin{array}{c|cc} A+B_uF-B_nC_y-B_uF_0C_y & B_n+B_uF_0 & B_w \\ \hline F-F_0C_y & F_0 & 0 \end{array} \right] \begin{bmatrix} x_K \\ y_1 \\ y_2 \end{bmatrix} \quad (5.14)$$

then  $\bar{\mathbf{K}}_{DF} \in \mathbf{K}_{DF}$  and  $\|F_\ell(F_\ell(\mathbf{G}_{DF}, \bar{\mathbf{K}}_{DF}), \Delta)\|_{MSP}^2 < \text{tr}(R)$

(c)  $(\nu_{DF}^*)^2 = \inf_{\mathbf{M}_F} \text{tr}(R)$

Proof in Appendix C.

**Remark 2.** Note that if  $A-B_nC_y$  is Schur in the networked DF synthesis, the existence of MS stabilizing controllers is governed by identical conditions as the networked FI problem. Moreover performing the networked FI synthesis to obtain  $F$  and  $F_0$  and forming  $\mathbf{K}_{DF}$  in (5.14) yields the same closed loop and cost for the networked DF problem as the networked FI scenario. This is the the equivalence of the networked FI and DF problems.

We now state several consequences of the networked FI/DF equivalence.

**Corollary 7.** Given the networked FI and DF problems, i.e. Problems 2 and 3, if  $A-B_nC_y$  Schur, then  $\nu_{FI}^* = \nu_{DF}^*$ .

**Corollary 8.**  $\mathbf{K}_{DF} \neq \emptyset$  if and only if  $(A, B_w, B_u, \Sigma)$  is MS stabilizable, and  $(A, B_w, B_u, \Sigma)$  is MS stabilizable if and only if there exist  $X \in \mathbf{S}_{++}^n$ ,  $Y \in \mathbf{R}^{m_u \times n}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  satisfying

$$\begin{bmatrix} X & (AX+B_uY) & B_wW \\ (AX+B_uY)^T & X & 0 \\ (B_wW)^T & 0 & W \end{bmatrix} \succ 0 \quad (5.15a)$$

$$\begin{bmatrix} Z & Y \\ Y^T & X \end{bmatrix} \succ 0 \quad (5.15b)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.15c)$$

Based on Lemma 3 and Corollary 7 we can conclude that optimal MS performance closed loop for the networked DF problem is  $H_{DF}^* = F_\ell(T_{DF}^*, \Delta) = F_\ell(F_\ell(G_{DF}, K_{DF}^*), \Delta)$  where

$$T_{DF}^* : \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A+B_u F^* & B_n+B_u F_0^* & B_w \\ \hline C_r+D_r F^* & D_{rn}+D_r F_0^* & D_{rw} \\ F^* & F_0^* & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (5.16)$$

$$K_{DF}^* : \begin{bmatrix} x_K^+ \\ u \end{bmatrix} = \left[ \begin{array}{c|cc} A+B_u F^*-B_n C_y-B_u F_0^* C_y & B_n+B_u F_0^* & B_w \\ \hline F^*-F_0^* C_y & F_0^* & 0 \end{array} \right] \begin{bmatrix} x_K \\ y_1 \\ y_2 \end{bmatrix} \quad (5.17)$$

The next results follow from the corresponding networked FI results in Propositions 3 and 2 using analogous arguments combined with the equivalence of the networked FI and DF problems.

**Proposition 3.** *If  $F$  is such that  $A + B_u F$  is Schur, then*

(a) *any stable, FDLTI  $T_{DF} = F_\ell(G_{DF}, K_{DF})$  can be realized as  $T_{DF} = F_\ell(U_{DF}, Q)$  where  $U_{DF}$  has the realization*

$$U_{DF} : \begin{bmatrix} \hat{x}^+ \\ r \\ z \\ n \\ w \end{bmatrix} = \left[ \begin{array}{c|ccc} A+B_u F & B_n & B_w & B_u \\ \hline C_r+D_r F & D_{rn} & \bar{D}_{rw} & D_r \\ F & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \\ q \end{bmatrix} \quad (5.18)$$

(b) *any MS stable  $H_{DF} = F_\ell(T_{DF}, \Delta)$  can be realized as  $H_{DF} = F_\ell(F_\ell(U_{DF}, Q), \Delta)$*

*for some stable, FDLTI system  $Q$ .*

It is easily verified that for any  $T_{DF} = F_\ell(G_{DF}, K_{DF})$  and  $Q$  where  $T_{DF} = F_\ell(U_{DF}, Q)$ , if  $K_{DF} \in \mathbf{K}_{DF}$  then  $Q$  must have the structure (5.19).

$$Q : \begin{bmatrix} x_Q^+ \\ q \end{bmatrix} = \left[ \begin{array}{c|cc} A_Q & B_{Q_1} & B_{Q_2} \\ \hline C_Q & D_{Q_1} & 0 \end{array} \right] \begin{bmatrix} x_Q \\ n \\ w \end{bmatrix} \quad (5.19)$$

with  $A_Q$  Schur.



**Proposition 4.** *Given a networked DF synthesis in Problem 3 where  $\mathbf{G}_{DF}$  in (5.2) meets the conditions of Lemmas 3 and 4, let  $F^*$ ,  $F_0^*$  be as given in Lemma 3, and  $\mathbf{K}_{DF}^*$  be as given in (5.17). Then*

$$\|F_\ell(F_\ell(\mathbf{G}_{DF}, \mathbf{K}_{DF}^*), \Delta)N^{\frac{1}{2}}\|_{MSP} \leq \|F_\ell(F_\ell(\mathbf{G}_{DF}, \mathbf{K}_{DF}), \Delta)N^{\frac{1}{2}}\|_{MSP} \quad (5.20)$$

for any  $N \succ 0$  and any  $\mathbf{K}_{DF} \in \mathbf{K}_{DF}$ .

**Remark 3.** *Although it can be shown the networked DF problem can be solved convexly without assuming  $A-B_nC_y$  is Schur, here we are only interested in the case with this assumption. First it simplifies the solution by guaranteeing the equivalence to the FI problem, and second we will see it is automatically met when we apply networked DF theory in Theorem 2.*

### 5.1.0.1 Existence of Equivalent Optimal $\mathcal{H}_2$ Performance Solutions

Before moving on to the dual problems of networked Full Control and Output Estimation, we briefly address the existence of  $\mathcal{H}_2$  performance synthesis problems for which the standard Riccati solutions, optimal control gains, and optimal performance cost coincide with the networked Full Information and Disturbance Feedforward problems.

Although we will see that these equivalent  $\mathcal{H}_2$  synthesis problems have little practical significance since they are formulated using an already known solution of the respective networked problems, the property that these problems exist and have equivalent solutions is of some theoretical interest and will be especially useful later on when we derive the optimal controller for the networked Output Feedback synthesis problem in Chapter 6.

**Problem 4** (Augmented Full Information  $\mathcal{H}_2$ ). *Let the augmented FI  $\mathcal{H}_2$  synthesis problem have generalized plant given by*

$$\hat{\mathbf{G}}_{FI}: \begin{bmatrix} x_G^+ \\ r_1 \\ r_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B_n & B_u \\ \hline C_r & D_{rn} & D_r \\ 0 & 0 & V^{\frac{1}{2}} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ u \end{bmatrix} \quad (5.21)$$

where  $V \in \mathbf{S}_+^{l_z}$ .

**Problem 5** (Augmented Disturbance Feedforward  $\mathcal{H}_2$ ). *Let the augmented DF  $\mathcal{H}_2$  synthesis problem have generalized plant given by*

$$\hat{\mathbf{G}}_{DF}: \begin{bmatrix} x_G^+ \\ r_1 \\ r_2 \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} A & B_n & B_u \\ \hline \begin{bmatrix} C_r \\ 0 \\ C_y \end{bmatrix} & \begin{bmatrix} D_{rn} \\ 0 \\ I \end{bmatrix} & \begin{bmatrix} D_r \\ V^{\frac{1}{2}} \\ 0 \end{bmatrix} \end{array} \right] \begin{bmatrix} x_G \\ n \\ u \end{bmatrix} \quad (5.22)$$

where  $V \in \mathbf{S}_+^{l_z}$ , and the matrix  $A - B_n C_y$  is Schur.

For Problem 4 and Problem 5, let the optimal closed loop  $\mathcal{H}_2$  performance be  $\gamma_{FI}^* = \gamma_{DF}^*$ , and under the assumptions of Lemma 3, let  $\hat{X}$  be the standard [66] Riccati equation solution and  $\hat{F}$  and  $\hat{F}_0$  be the optimal control gains.

**Lemma 5.** *Given the Networked FI or DF synthesis in Problem 2 or 3 with the conditions of Lemma 3, let  $X^*$ ,  $R^*$ ,  $F^*$ ,  $F_0^*$ , and  $\nu_{FI}^* = \nu_{DF}^*$  be the optimal solution in Lemma 3, and let  $V = \Sigma \circ (B_w^T X^* B_w + D_{rw}^T D_{rw})$  in Problem 4 or 5. Then  $X^* = \hat{X}$ , the optimal cost  $\nu_{FI}^* = \gamma_{FI}^* = \nu_{DF}^* = \gamma_{DF}^*$ , and the optimal control gains  $F^* = \hat{F}$  and  $F_0^* = \hat{F}_0$ .*

Proof in Appendix C.

## 5.2 Networked Full Control and Output Estimation

In this section we investigate networked extensions of the classical FC and OE problems. The networked FC and OE scenarios are depicted in Figure 5.2a and 5.2b respectively. Notice that in each case we consider the sensor link subject to stochastic uncertainty.

We point out that these problems are the dual problems of the networked FI and DF problems we studied in Section 5.1. As a consequence, all results in this section follow those in Section 5.1 by duality, and we will give more limited discussion since much of it would be effectively redundant.

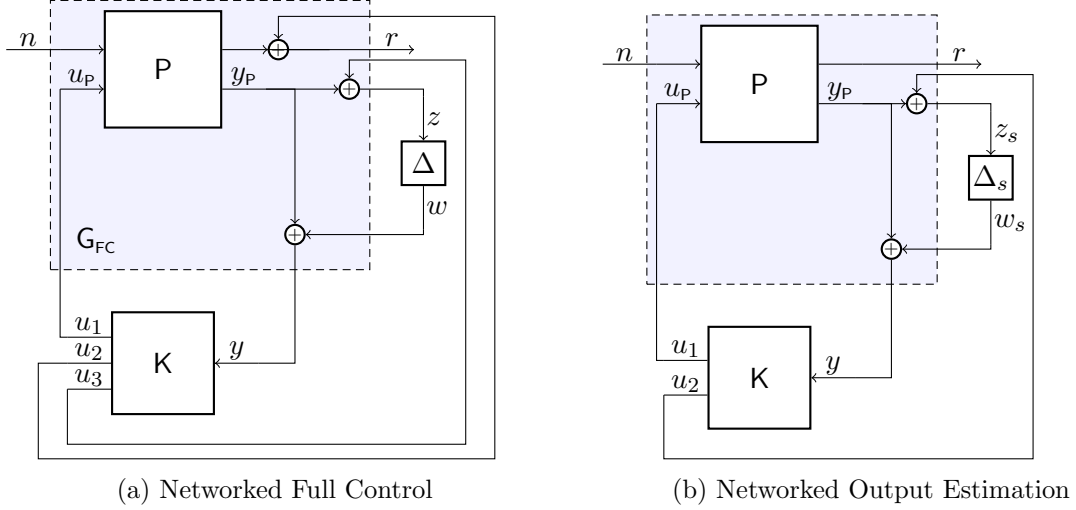


Figure 5.2: Networked FC and OE Synthesis Setups

**Problem 6** (Networked Full Control). *Let the networked FC synthesis problem have generalized plant given by*

$$G_{FC}: \begin{bmatrix} x_G^+ \\ r \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_n & 0 & I & 0 & 0 \\ \hline C_r & D_{rn} & 0 & 0 & I & 0 \\ C_z & D_{zn} & 0 & 0 & 0 & I \\ C_y & D_n & I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (5.23)$$

with sensor uncertainty represented by  $\Delta(k) : z(k) \mapsto w(k)$  having  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$ . Let  $\mathbf{K}_{FC}$  be the set of all FDLTI controllers making  $H_{FC} = F_\ell(F_\ell(G_{FC}, \mathbf{K}_{FC}), \Delta)$  MS stable which are strictly proper from  $y$  to  $u_3$ , and let  $\nu_{FC}^* \triangleq \inf_{\mathbf{K}_{FC} \in \mathbf{K}_{FC}} \|H_{FC}\|_{MSP}$ .

**Problem 7** (Networked Output Estimation (OE)). *Let the networked OE synthesis problem have generalized plant given by*

$$G_{OE}: \begin{bmatrix} x_G^+ \\ r \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_n & 0 & B_u & 0 \\ \hline C_r & D_{rn} & 0 & I & 0 \\ C_z & D_{zn} & 0 & 0 & I \\ C_y & D_n & I & 0 & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w \\ u_1 \\ u_2 \end{bmatrix} \quad (5.24)$$

with sensor uncertainty represented by  $\Delta(k) : z(k) \mapsto w(k)$  with  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$ . Let  $\mathbf{K}_{\text{OE}}$  be the set of all FDLTI controllers making  $\mathbf{H}_{\text{OE}} = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{G}_{\text{OE}}, \mathbf{K}_{\text{OE}}), \Delta)$  MS stable which are strictly proper from  $y$  to  $u_2$ , and let  $\nu_{\text{OE}}^* \triangleq \inf_{\mathbf{K}_{\text{OE}} \in \mathbf{K}_{\text{OE}}} \|\mathbf{H}_{\text{OE}}\|_{\text{MSP}}$ .

**Definition 8.** Given  $\mathbf{G}_{\text{FC}}$  in (5.23) or  $\mathbf{G}_{\text{OE}}$  in (5.24) with  $\text{cov}(\Delta(k)\mathbf{1}) = \Sigma \in \mathbf{B}_+^\eta$  let  $\mathbf{M}_L$  be the feasible set of all  $X \in \mathbf{S}_+^n$ ,  $Y \in \mathbf{R}^{n \times l_y}$ ,  $J \in \mathbf{R}^{l_r \times l_y}$ ,  $R \in \mathbf{S}_+^{l_r}$ ,  $Z \in \mathbf{S}_+^{l_z}$ ,  $W \in \mathbf{B}_+^\eta$  satisfying the LMIs (5.25a)-(5.25d).

$$\begin{bmatrix} X & (XA + YC_y)^T & (C_r + JC_y)^T & (WC_y)^T \\ XA + YC_y & X & 0 & 0 \\ C_r + JC_y & 0 & I & 0 \\ WC_y & 0 & 0 & W \end{bmatrix} \succ 0 \quad (5.25a)$$

$$\begin{bmatrix} R & (XB_n + YD_n)^T & (D_{rn} + JD_n)^T & (WD_n)^T \\ XB_n + YD_n & X & 0 & 0 \\ D_{rn} + JD_n & 0 & I & 0 \\ WD_n & 0 & 0 & W \end{bmatrix} \succ 0 \quad (5.25b)$$

$$\begin{bmatrix} Z & Y^T & J^T \\ Y & X & 0 \\ J & 0 & I \end{bmatrix} \succ 0 \quad (5.25c)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.25d)$$

As with the FI problem, we begin by showing that the existence of a MS stabilizing controller for the networked Full Control synthesis in Problem 6 is equivalent to an LMI feasibility problem, and that we can obtain a static controller with MS performance arbitrarily close to the optimal from the feasible set.

**Lemma 6.** Given Problem 6, let  $\bar{\mathbf{K}}_{\text{FC}} \subseteq \mathbf{K}_{\text{FC}}$  be the admissible static controllers, i.e. all  $\bar{\mathbf{K}}_{\text{FC}} \in \mathbf{K}_{\text{FC}}$  with a realization (5.26), and let  $\mathbf{M}_{\text{L}}$  be as given in Definition 8. Then

$$\bar{\mathbf{K}}_{\text{FC}} : u = \begin{bmatrix} L \\ L_0 \\ 0 \end{bmatrix} y \quad (5.26)$$

$$(a) \mathbf{K}_{\text{FC}} \neq \emptyset \Leftrightarrow \bar{\mathbf{K}}_{\text{FC}} \neq \emptyset \Leftrightarrow \mathbf{M}_{\text{L}} \neq \emptyset$$

$$(b) (X, Y, J, R, Z, W) \in \mathbf{M}_{\text{L}} \Rightarrow \bar{\mathbf{K}}_{\text{FC}} = \begin{bmatrix} X^{-1}Y^T \\ J^T \\ 0 \end{bmatrix} \in \bar{\mathbf{K}}_{\text{FC}} \text{ and } \|F_\ell(F_\ell(\mathbf{G}_{\text{FC}}, \bar{\mathbf{K}}_{\text{FC}}), \Delta)\|_{\text{MSP}}^2 < \text{tr}(R)$$

$$(c) (\nu_{\text{FC}}^*)^2 = \inf_{\mathbf{M}_{\text{L}}} \text{tr}(R)$$

Analogous to the networked FI case, we can show that the existence of a MS stabilizing controller for the networked Full Control problem is equivalent to  $(A, C_z, C_y, \Sigma)$  MS detectable, which moreover is equivalent to the feasibility of a simplified set of LMIs.

**Corollary 9.**  $\mathbf{K}_{\text{FC}} \neq \emptyset$  if and only if  $(A, C_z, C_y, \Sigma)$  is MS detectable, and  $(A, C_z, C_y, \Sigma)$  is MS detectable if and only if there exist  $X \in \mathbf{S}_+^n$ ,  $Y \in \mathbf{R}^{n \times l_y}$ ,  $Z \in \mathbf{S}_+^{l_z}$ , and  $W \in \mathbf{B}_+^\eta$  such that

$$\begin{bmatrix} X & (XA + YC_y)^T & (WC_y)^T \\ XA + YC_y & X & 0 \\ WC_y & 0 & W \end{bmatrix} \succ 0 \quad (5.27a)$$

$$\begin{bmatrix} Z & Y^T \\ Y & X \end{bmatrix} \succ 0 \quad (5.27b)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.27c)$$

The static controller  $\bar{\mathbf{K}}_{\text{FC}} = \begin{bmatrix} X^{-1}Y^T \\ J^T \\ 0 \end{bmatrix} \in \bar{\mathbf{K}}_{\text{FC}}$  for any feasible  $X$  and  $Y$ .

As in Section 5.1 we next investigate the limit points of MS stabilizing control gains.

**Lemma 7.** Given Problem 6 with  $(A, C_z, C_y, \Sigma)$  MS detectable,  $D_n D_n^T + \Sigma \circ D_{zn} D_{zn}^T \succ 0$ ,  $D_n B_n^T = 0$ , and  $(A, B_n)$  stabilizable,

(a) there exists a unique  $X^* \succeq 0$  and  $R^* \succeq 0$ :

$$X^* = AX^*A^T + B_n B_n^T - (AX^*C_y^T)S^{-1}(C_y X^*A^T) \quad (5.28a)$$

$$R^* = C_r X^*C_r^T + D_{rn}D_{rn}^T - (C_r X^*C_y^T + D_{rn}D_{rn}^T)S^{-1}(C_y X^*C_r^T + D_n D_{rn}^T) \quad (5.28b)$$

$$S = C_y X^*C_y^T + D_n D_n^T + \Sigma \circ (C_z X^*C_z^T + D_{zn}D_{zn}^T) \quad (5.28c)$$

and controller gains

$$L^* = -(AX^*C_y^T)S^{-1} \quad (5.29a)$$

$$L_0^* = -(C_r X^*C_y^T + D_{rn}D_{rn}^T)S^{-1} \quad (5.29b)$$

such that  $\|\mathbf{H}_{FC}\|_{MSP} = \nu_{FC}^* = \text{tr}(R^*)^{\frac{1}{2}}$ .

(b) for any  $X(0) \succeq 0$ ,  $X^* = \lim_{k \rightarrow \infty} X(k)$  where

$$S(k) = C_y X(k)C_y^T + D_n D_n^T + \Sigma \circ (C_z X(k)C_z^T + D_{zn}D_{zn}^T) \quad (5.30a)$$

$$X(k+1) = AX(k)A^T + B_n B_n^T - (AX(k)C_y^T)S^{-1}(C_y X(k)A^T) \quad (5.30b)$$

Based on Lemma 7 we can conclude that optimal MS performance closed loop for the networked FC problem is  $\mathbf{H}_{FC}^* = F_\ell(\mathbf{T}_{FC}^*, \Delta) = F_\ell(F_\ell \mathbf{G}_{FC}, \mathbf{K}_{FC}^*, \Delta)$  where

$$\mathbf{T}_{FC}^* : \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A + L^*C_y & B_n + L^*D_n & L^* \\ \hline C_r + L_0^*C_y & D_{rn} + L_0^*D_n & L_0^* \\ C_z & D_{zn} & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (5.31)$$

$$\mathbf{K}_{FC}^* : u = \begin{bmatrix} L^* \\ L_0^* \\ 0 \end{bmatrix} y \quad (5.32)$$

**Proposition 5.** If  $L$  is such that  $A + LC_y$  is Schur, then

(a) any stable, FDLTI  $T_{FC} = F_\ell(G_{FC}, K_{FC})$  can be realized as  $T_{FC} = F_\ell(U_{FC}, Q)$  where  $U_{FC}$  has the realization

$$U_{FC} : \begin{bmatrix} x_G^+ \\ r \\ z \\ s \end{bmatrix} = \left[ \begin{array}{c|ccc} A+LC_y & B_n+LD_n & L & 0 & 0 \\ \hline C_r & D_{rn} & 0 & I & 0 \\ C_z & D_{zn} & 0 & 0 & I \\ C_y & D_n & I & 0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \\ q_1 \\ q_2 \end{bmatrix} \quad (5.33)$$

(b) any MS stable  $H_{FC} = F_\ell(T_{FC}, \Delta)$  can be realized as  $H_{FC} = F_\ell(F_\ell(U_{FC}, Q), \Delta)$

for some stable, FDLTI system  $Q$ .

It is easily verified that for any  $T_{FC} = F_\ell(G_{FC}, K_{FC})$  and  $Q$  where  $T_{FC} = F_\ell(U_{FC}, Q)$ , if  $K_{FC} \in \mathbf{K}_{FC}$  then  $Q$  must have the structure (5.34).

$$Q : \begin{bmatrix} x_Q^+ \\ q_1 \\ q_2 \end{bmatrix} = \left[ \begin{array}{c|c} A_Q & B_Q \\ \hline C_{Q_1} & D_{Q_1} \\ C_{Q_2} & 0 \end{array} \right] \begin{bmatrix} x_Q \\ y \end{bmatrix} \quad (5.34)$$

with  $A_Q$  Schur.

Before addressing the networked OE problem, as with the networked FI scenario, the optimal control in Lemma 7 is in fact optimal for a set of networked FC closed loops.

**Proposition 6.** Given a networked FC synthesis in Problem 3 where  $G_{FC}$  in (5.23) meets the conditions of Lemmas 7, let  $L^*$ ,  $L_0^*$  be as given in (5.29), and  $K_{FC^*}$  be as given in (5.32). Then

$$\|N^{\frac{1}{2}} F_\ell(F_\ell(G_{FC}, K_{FC}^*), \Delta)\|_{MSP} \leq \|N^{\frac{1}{2}} F_\ell(F_\ell(G_{FC}, K_{FC}), \Delta)\|_{MSP} \quad (5.35)$$

for any  $N \succ 0$  and any  $K_{FC} \in \mathbf{K}_{FC}$ .

Next we move on to the networked OE problem. We will see that just as with the networked FI and DF problems studied in Section 5.1, the networked OE problem is equivalent to the networked FC problem given a standard assumption, and moreover is the dual of the networked DF problem .

**Lemma 8.** *Given Problem 7 with  $A-B_u C_r$  Schur, let  $\mathbf{M}_L$  be as given in Definition 8. Then*

(a)  $\mathbf{K}_{OE} \neq \emptyset \Leftrightarrow \mathbf{M}_L \neq \emptyset$

(b) if  $(X, Y, J, R, Z, W) \in \mathbf{M}_L$ ,  $L = X^{-1}Y^T$ ,  $L_0 = J^T$  and

$$\bar{\mathbf{K}}_{OE}: \begin{bmatrix} x_K^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|c} \frac{A+LC_y-B_u C_r-B_u L_0 C_y}{C_r+L_0 C_y} & \frac{L-B_u L_0}{L_0} \\ \hline & 0 \end{array} \right] \begin{bmatrix} x_K \\ y \end{bmatrix} \quad (5.36)$$

then  $\bar{\mathbf{K}}_{OE} \in \mathbf{K}_{OE}$  and  $\|F_\ell(F_\ell(\mathbf{G}_{OE}, \bar{\mathbf{K}}_{OE}), \Delta)\|_{MSP}^2 < \text{tr}(R)$

(c)  $(\nu_{OE}^*)^2 = \inf_{\mathbf{M}_L} \text{tr}(R)$

**Remark 4.** *As with the FI/DF case we have the the FC and OE problems are equivalent when  $A-C_r B_u$  is Schur, i.e. the existence of MS stabilizing controllers is governed by identical conditions and the networked OE synthesis can be carried out using the result of a networked FC synthesis to obtain the same closed loop and cost.*

We next state several consequences of this equivalence.

**Corollary 10.** *Given Problem 6 and Problem 7 with  $A-C_r B_u$  Schur,  $\nu_{FC}^* = \nu_{OE}^*$ .*

**Corollary 11.**  $\mathbf{K}_{OE} \neq \emptyset$  if and only if  $(A, C_z, C_y, \Sigma)$  is MS detectable, and  $(A, C_z, C_y, \Sigma)$  is MS detectable if and only if there exist  $X \in \mathbf{S}_+^n$ ,  $Y \in \mathbf{R}^{n \times l_y}$ ,  $Z \in \mathbf{S}_+^{l_z}$ , and  $W \in \mathbf{B}_+^\eta$  such that

$$\begin{bmatrix} X & (XA + YC_y)^T & (WC_y)^T \\ XA + YC_y & X & 0 \\ WC_y & 0 & W \end{bmatrix} \succ 0 \quad (5.37a)$$

$$\begin{bmatrix} Z & Y^T \\ Y & X \end{bmatrix} \succ 0 \quad (5.37b)$$

$$W - \Sigma \circ Z \succ 0 \quad (5.37c)$$



Based on Lemma 7 and Corollary 10 we can conclude that optimal MS performance closed loop for the networked OE problem is  $H_{OE}^* = F_\ell(T_{OE}^*, \Delta) = F_\ell(F_\ell(G_{FC}, K_{FC}^*), \Delta)$  where

$$T_{OE}^* : \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A+L^*C_y & B_n+L^*D_n & L^* \\ \hline C_r+L_0^*C_y & D_{rn}+L_0^*D_n & L_0^* \\ \bar{C}_z & D_{zn} & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (5.38)$$

$$\bar{K}_{OE} : \begin{bmatrix} x_K^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|c} A+L^*C_y-B_uC_r-B_uL_0C_y & L^*-B_uL_0^* \\ \hline C_r+L_0^*C_y & L_0^* \\ C_y & 0 \end{array} \right] \begin{bmatrix} x_K \\ y \end{bmatrix} \quad (5.39)$$

**Proposition 7.** *If  $L$  is such that  $A + LC_y$  is Schur, then*

(a) *any stable, FDLTI  $T_{OE} = F_\ell(G_{OE}, K_{OE})$  can be realized as  $T_{OE} = F_\ell(U_{OE}, Q)$  where  $U_{OE}$  has the realization*

$$U_{OE} : \begin{bmatrix} x_G^+ \\ r \\ z \\ s \end{bmatrix} = \left[ \begin{array}{c|ccc} A+LC_y & B_n+LD_n & L & 0 & 0 \\ \hline C_r & D_{rn} & 0 & I & 0 \\ C_z & D_{zn} & 0 & 0 & I \\ C_y & D_n & I & 0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \\ q_1 \\ q_2 \end{bmatrix} \quad (5.40)$$

(b) *any MS stable  $H_{OE} = F_\ell(T_{OE}, \Delta)$  can be realized as  $H_{OE} = F_\ell(F_\ell(U_{OE}, Q), \Delta)$*

*for some stable, FDLTI system  $Q$ .*

It is easily verified that for any  $T_{OE} = F_\ell(G_{OE}, K_{OE})$  and  $Q$  where  $T_{OE} = F_\ell(U_{OE}, Q)$ , if  $K_{OE} \in \mathbf{K}_{OE}$  then  $Q$  must have the structure (5.41).

$$Q : \begin{bmatrix} x_Q^+ \\ q_1 \\ q_2 \end{bmatrix} = \left[ \begin{array}{c|c} A_Q & B_Q \\ \hline C_{Q_1} & D_{Q_1} \\ C_{Q_2} & 0 \end{array} \right] \begin{bmatrix} x_Q \\ y \end{bmatrix} \quad (5.41)$$

with  $A_Q$  Schur.

**Proposition 8.** *Given a networked OE synthesis in Problem 3 where  $\mathbf{G}_{\text{OE}}$  in (5.24) meets the conditions of Lemmas 7 and 8, let  $L^*$ ,  $L_0^*$  be as given in (5.29), and  $\mathbf{K}_{\text{OE}^*}$  be as given in (5.39). Then*

$$\|N^{\frac{1}{2}} \mathbb{F}_\ell(\mathbb{F}_\ell(\mathbf{G}_{\text{OE}}, \mathbf{K}_{\text{OE}^*}), \Delta)\|_{\text{MSP}} \leq \|N^{\frac{1}{2}} \mathbb{F}_\ell(\mathbb{F}_\ell(\mathbf{G}_{\text{OE}}, \mathbf{K}_{\text{OE}}), \Delta)\|_{\text{MSP}} \quad (5.42)$$

for any  $N \succ 0$  and any  $\mathbf{K}_{\text{OE}} \in \mathbf{K}_{\text{OE}}$ .

### 5.2.0.2 Existence of Equivalent Optimal $\mathcal{H}_2$ Performance Solutions

Again, as with the networked FI and DF problems, we briefly address the existence of  $\mathcal{H}_2$  performance synthesis problems for which the standard Riccati solutions, optimal control gains, and optimal performance cost coincide with the networked FC and OE problems.

**Problem 8** (Augmented Full Control  $\mathcal{H}_2$ ). *Let the augmented FC  $\mathcal{H}_2$  synthesis problem have generalized plant*

$$\mathbf{G}_{\text{FC}}: \begin{bmatrix} x_G^+ \\ r \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} A & \begin{bmatrix} B_n & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_r & \begin{bmatrix} D_{rn} & 0 \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_y & \begin{bmatrix} D_n & W^{\frac{1}{2}} \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} \right] \begin{bmatrix} x_G \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix} \quad (5.43)$$

where  $W \in \mathbf{S}_+^{l_z}$ .

**Problem 9** (Augmented Output Estimation  $\mathcal{H}_2$ ). *Let the augmented OE  $\mathcal{H}_2$  synthesis problem have generalized plant*

$$\mathbf{G}_{\text{OE}}: \begin{bmatrix} x_G^+ \\ r \\ y \end{bmatrix} = \left[ \begin{array}{c|cc} A & \begin{bmatrix} B_n & 0 \end{bmatrix} & B_u \\ \hline C_r & \begin{bmatrix} D_{rn} & 0 \end{bmatrix} & I \\ C_y & \begin{bmatrix} D_n & W^{\frac{1}{2}} \end{bmatrix} & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \\ u_1 \\ u_2 \end{bmatrix} \quad (5.44)$$

where  $W \in \mathbf{S}_+^{l_z}$ , and the matrix  $A - B_u C_r$  is Schur.

For Problem 8 and Problem 9, let the optimal closed loop  $\mathcal{H}_2$  performance be  $\gamma_{FC}^* = \gamma_{OE}^*$ , and under the assumptions of Lemma 7, let  $\hat{Y}$  be the standard [66] Riccati equation solution and  $\hat{L}$  and  $\hat{L}_0$  be the optimal control gains.

**Lemma 9.** *Given the Networked FC or OE synthesis in Problem 6 or 7 with the conditions of Lemma 7, let  $X^*$ ,  $R^*$ ,  $L^*$ ,  $L_0^*$  and  $\nu_{FC}^* = \nu_{DF}^*$  be the optimal solution in Lemma 7, and let  $W = \Sigma \circ (C_z X^* C_z^T + D_{zn} D_{zn}^T)$ . Then  $X^* = \hat{Y}$ , the optimal cost  $\nu_{FI}^* = \gamma_{FI}^* = \nu_{DF}^* = \gamma_{DF}^*$ , and the optimal control gains  $L^* = \hat{L}$  and  $L_0^* = \hat{L}_0$ .*

## CHAPTER 6. NETWORKED OUTPUT FEEDBACK

In this chapter we investigate a networked extension of the standard output feedback synthesis. The networked OF scenario is shown in Figure 6.1. Notice that we consider both the actuator link and sensor link subject to stochastic uncertainty. We formally define this problem next.

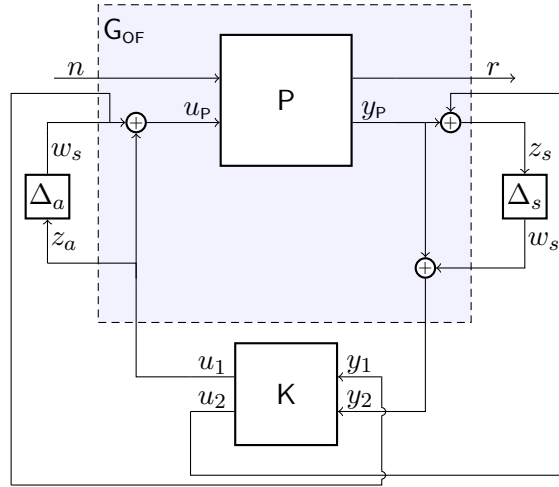


Figure 6.1: Networked Output Feedback

**Problem 10** (Networked Output Feedback). *Let the networked OF synthesis problem have generalized plant given by*

$$G_{\text{OF}}: \begin{bmatrix} x_G^+ \\ r \\ z_s \\ z_a \\ y_s \\ y_a \end{bmatrix} = \begin{bmatrix} A & B_n & \begin{bmatrix} 0 & B_w \end{bmatrix} & \begin{bmatrix} 0 & B_u \end{bmatrix} \\ C_r & D_{rn} & \begin{bmatrix} 0 & D_{rw} \end{bmatrix} & \begin{bmatrix} 0 & D_r \end{bmatrix} \\ \begin{bmatrix} C_z \end{bmatrix} & \begin{bmatrix} D_{zn} \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} \\ 0 & 0 & \begin{bmatrix} 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} \\ \begin{bmatrix} C_y \end{bmatrix} & \begin{bmatrix} D_n \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \\ 0 & 0 & \begin{bmatrix} 0 & I \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w_s \\ w_a \\ u_s \\ u_a \end{bmatrix} \quad (6.1)$$

with uncertainty  $\Delta(k) = \text{diag}(\Delta_s(k), \Delta_a(k)) : z(k) \mapsto w(k)$  having  $\Sigma = \text{diag}(\Sigma_s, \Sigma_a) = \text{cov}(\Delta(k)\mathbf{1}) \in \mathbf{B}_+^n$ . Let  $\mathbf{K}_{\text{OF}}$  be the set of all FDLTI controllers with a realization (6.2) making  $\mathbf{H}_{\text{OF}} = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{G}_{\text{OF}}, \mathbf{K}_{\text{OF}}), \Delta)$  MS stable, and  $\nu_{\text{OF}}^* \triangleq \inf_{\mathbf{K} \in \mathbf{K}_{\text{OF}}} \|\mathbf{H}_{\text{OF}}\|_{\text{MSP}}$ .

$$\mathbf{K}_{\text{OF}} : \begin{bmatrix} x_{\mathbf{K}}^+ \\ u_s \\ u_a \end{bmatrix} = \left[ \begin{array}{c|cc} A_{\mathbf{K}} & B_{\mathbf{K}_s} & B_{\mathbf{K}_a} \\ \hline B_{\mathbf{K}_s} & 0 & 0 \\ B_{\mathbf{K}_a} & D_{\mathbf{K}_{as}} & 0 \end{array} \right] \begin{bmatrix} x_{\mathbf{K}} \\ y_s \\ y_a \end{bmatrix} \quad (6.2)$$

Note that the setup in Figure 6.1 is equivalent the setup of Figure 4.2 in Chapter 4. Since we assume each channel mean is known, the unknown information provided by  $u_p(k-1) = M_a u_1(k-1) + w_a(k-1)$  is  $w_a(k-1)$ . Similarly, an input to  $y_p(k)$  can be replaced an input into  $z_s(k)$ . Finally, we can assume without loss of generality identity channel means.

As discussed therein, the problem of Section 4.2 cannot be solved convexly by adapting LMIs of Theorem 1 for synthesis using the standard change of coordinates [64] since doing so does not eliminate all nonlinear coupling between the problem variables. The networked OF synthesis in Problem 10 presents the same difficulty. However, we will show that the optimal solution to Problem 10 can be obtained by solving a sequence of the special problems we studied in Sections 5.1 and 5.2. Then we will see that this solution can be adapted to provide the solution to the optimal MS performance synthesis problem in Chapter 4 as a special case. We begin by building up some preliminary results.

## 6.1 Suboptimal Networked Output Feedback

We first focus on nominal stabilization of (6.1) and show how any stabilizing controller  $\mathbf{K}_{\text{OF}}$  and any stable  $\mathbf{T}_{\text{OF}} = \mathbf{F}_\ell(\mathbf{G}_{\text{OF}}, \mathbf{K}_{\text{OF}})$  can be realized using standard Youla parameterization techniques. This then allows us to realize any MS stable  $\mathbf{H}_{\text{OF}} = \mathbf{F}_\ell(\mathbf{T}_{\text{OF}}, \Delta)$  by Corollary 4.

**Proposition 9.** *If  $F$  and  $L$  are such that  $A+B_u F$  and  $A+LC_y$  are Schur, then*

(a) any FDLTI causal controller that internally stabilizes  $G_{\text{OF}}$  in (6.1) can be obtained as the transfer function matrix of  $K_{\text{OF}} = F_{\ell}(J, Q)$  where  $J$  has the realization

$$J: \begin{bmatrix} \hat{x}^+ \\ u_s \\ u_a \\ s \\ w_a \end{bmatrix} = \left[ \begin{array}{c|cc} A+LC_y+B_uF & \begin{bmatrix} -L & B_w \end{bmatrix} & \begin{bmatrix} 0 & B_u \end{bmatrix} \\ \hline \begin{bmatrix} -C_z \\ F \\ -C_y \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right] \begin{bmatrix} \hat{x} \\ y_s \\ y_a \\ q_1 \\ q_2 \end{bmatrix} \quad (6.3)$$

(b) any stable  $T_{\text{OF}} = F_{\ell}(G_{\text{OF}}, K_{\text{OF}})$  can be obtained via a feedback interconnection:

$$T_{\text{OF}} = F_{\ell}(G_{\text{OF}}, F_{\ell}(J, Q)) \quad (6.4)$$

(c) any MS stable  $H_{\text{OF}} = F_{\ell}(T_{\text{OF}}, \Delta)$  can be obtained via a feedback interconnection:

$$H_{\text{OF}} = F_{\ell}(F_{\ell}(G_{\text{OF}}, F_{\ell}(J, Q)), \Delta) \quad (6.5)$$

for some stable, FDLTI system  $Q$ .

Proof in Appendix D

Noting the structure of  $J$  in (6.3), it is easily verified that for any  $K_{\text{OF}} = F_{\ell}(J, Q)$  we have  $D_K = D_Q$ . Therefore, recalling (6.2) in Problem 10, any  $Q$  which will realize  $K_{\text{OF}} \in \mathbf{K}_{\text{OF}}$  must have the structure (6.6).

$$Q: \begin{bmatrix} x_Q^+ \\ q_1 \\ q_2 \end{bmatrix} = \left[ \begin{array}{c|cc} A_Q & B_{Q_1} & B_{Q_2} \\ \hline C_{Q_1} & 0 & 0 \\ C_{Q_2} & D_{Q_{21}} & 0 \end{array} \right] \begin{bmatrix} x_Q \\ s \\ w_a \end{bmatrix} \quad (6.6)$$

Next we show how  $G_{\text{OF}}$  in (6.1) is equivalent to the interconnection of two simpler subsystems: one corresponding to a FC problem and one corresponding to a DF problem. This then allows us to realize any stable closed loop  $T_{\text{OF}}$  as the interconnection of FC and DF closed loops.

**Proposition 10.** For any  $F$  and  $L$  be such that  $A+B_uF$  and  $A+LC_y$  are Schur, and any  $L_0$  of compatible dimensions:

(a)  $G_{OF}$  in (6.1) is equal to the interconnection of  $G_{FC}$  in (6.7) and  $G_{DF}$  in (6.8) shown in Figure 6.2, where  $r(k) = r_e(k) + r_{\hat{x}}(k)$ , and  $z_s(k) = z_{s_e}(k) + z_{s_{\hat{x}}}(k)$  as shown.

$$G_{FC}: \begin{bmatrix} e^+ \\ r_e \\ z_{s_e} \\ s \end{bmatrix} = \left[ \begin{array}{c|ccc} A+LC_y & B_n+LD_n & L & \\ \hline C_r+L_0C_y & D_{rn}+L_0D_n & L_0 & \\ C_z & D_{zn} & 0 & \\ C_y & D_n & I & \end{array} \right] \begin{bmatrix} e \\ n \\ w_s \end{bmatrix} \quad (6.7)$$

$$G_{DF}: \begin{bmatrix} \hat{x}^+ \\ r_{\hat{x}} \\ z_{s_{\hat{x}}} \\ z_a \\ y_s \\ y_a \end{bmatrix} = \left[ \begin{array}{c|ccccc} A & -L & B_w & 0 & B_u \\ \hline C_r & -L_0 & D_{rw} & 0 & D_r \\ C_z & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ C_y & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{array} \right] \begin{bmatrix} \hat{x} \\ s \\ w_a \\ u_s \\ u_a \end{bmatrix} \quad (6.8)$$

(b) any stable  $T_{OF} = F_{\ell}(G_{OF}, K_{OF})$  can be recovered as the interconnection of  $G_{FC}$  and  $F_{\ell}(G_{DF}, F_{\ell}(J, Q))$  shown in Figure 6.2, where  $K_{OF} = F_{\ell}(J, Q)$  as in Proposition 9.

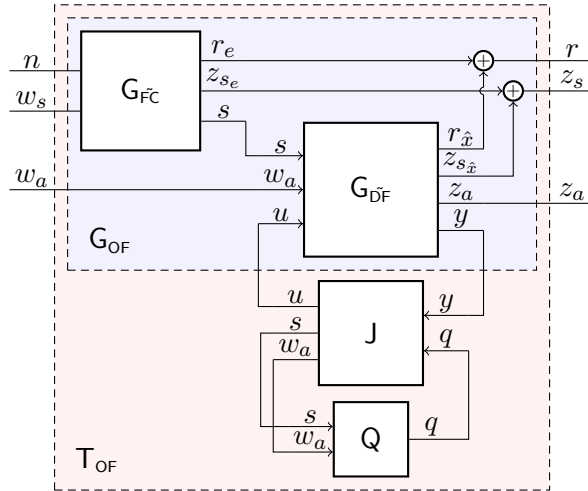


Figure 6.2: Structure of any stable closed loop  $T_{OF}$

Proof in Appendix D

Now, based on Proposition 10, we show a modified structure for any stable  $T_{OF} = F_{\ell}(G_{OF}, K_{OF})$  which will be more useful.

**Proposition 11.** For any  $F$  and  $L$  be such that  $A+B_uF$  and  $A+LC_y$  are Schur, and any  $L_0$  of compatible dimensions, any stable  $T_{OF} = F_\ell(G_{OF}, K_{OF})$  can be realized as the interconnection of  $G_{\bar{F}C}$ ,  $U_{DF}$ ,  $Q_{11}$ ,  $Q_{12}$ , and  $Q_2$  shown in Figure 6.3, where  $G_{\bar{F}C}$  and  $U_{DF}$  are given in (6.7) and (6.9) respectively, and  $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$  has properties given in Proposition 9 with  $q_{1_s} = Q_{11}s$ ,  $q_{1_w} = Q_{12}w_a$ , and  $q_2 = Q_2 \begin{bmatrix} s \\ w_a \end{bmatrix}$ .

$$U_{DF}: \begin{bmatrix} \hat{x}^+ \\ r_{\hat{x}} \\ z_a \\ s \\ w_a \end{bmatrix} = \begin{bmatrix} A+B_uF & -L & B_w & B_u \\ \hline C_r+D_rF & -L_0 & D_{rw} & D_r \\ F & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ s \\ w_a \\ q_2 \end{bmatrix} \quad (6.9)$$

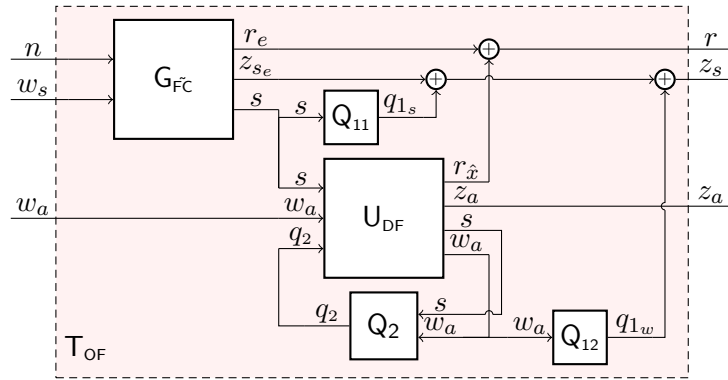


Figure 6.3: Equivalent structure of any stable closed loop  $T_{OF}$

Proof in Appendix D

For convenience, we define a notation which will allow us to refer explicitly to a specific parameterization of  $T_{OF}$ .

**Notation 1.** Given a stable  $T_{OF} = F_\ell(G_{OF}, K_{OF})$  and a realization of  $T_{OF}$  parameterized by  $F$ ,  $L$ ,  $L_0$ , and  $Q$ , let  $T_{OF} = \mathcal{T}_{OF}(F, L, L_0, Q)$ .

**Definition 9.** For Problem 10 let  $\mathbf{H}_{OF} = \{F_\ell(F_\ell(G_{OF}, K_{OF}), \Delta) \mid K_{OF} \in \mathbf{K}_{OF}\}$ .

From here on we will progressively narrow down  $\mathbf{H}_{OF}$  to eventually isolate an optimal closed loop. Next we will see that we can restrict our attention to  $H_{OF} = F_\ell(T_{OF}, \Delta) \in \mathbf{H}_{OF}$  where  $T_{OF} = \mathcal{T}_{OF}(F, L, L_0, Q)$  with  $Q_{12} = 0$ .



**Lemma 10.** Given  $\mathbf{H}_{\text{OF}} = F_\ell(\mathcal{T}_{\text{OF}}(F, L, L_0, \mathbf{Q}), \Delta) \in \mathbf{H}_{\text{OF}}$ , let  $\bar{\mathbf{H}}_{\text{OF}} = F_\ell(\mathcal{T}_{\text{OF}}(F, L, L_0, \bar{\mathbf{Q}}), \Delta)$  where  $\bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_{11} & 0 \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$ . Then  $\bar{\mathbf{H}}_{\text{OF}} \in \mathbf{H}_{\text{OF}}$  and  $\|\bar{\mathbf{H}}_{\text{OF}}\|_{\text{MSP}} \leq \|\mathbf{H}_{\text{OF}}\|_{\text{MSP}}$ .

Proof in Appendix D. We use this to define a subset of  $\mathbf{H}_{\text{OF}}$ .

**Definition 10.** Let  $\mathbf{H}'_{\text{OF}} = \{\mathbf{H}_{\text{OF}} = F_\ell(\mathcal{T}_{\text{OF}}(F, L, L_0, \mathbf{Q}), \Delta) \in \mathbf{H}_{\text{OF}} \mid \mathbf{Q}_{12} = 0\}$ .

By Lemma 10 there is no loss of generality in restricting our attention to  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$ . Note any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  has the structure in Figure 6.4 where  $\tilde{\mathbf{U}}_{\text{FC}}$  has realization (6.10).

$$\tilde{\mathbf{U}}_{\text{FC}}: \begin{bmatrix} e^+ \\ r_e \\ z_s \\ s \end{bmatrix} = \begin{bmatrix} A+LC_y & B_n+LD_n & L & 0 \\ C_r+L_0C_y & D_{rn}+L_0D_n & L_0 & 0 \\ C_z & D_{zn} & 0 & I \\ C_y & D_n & I & 0 \end{bmatrix} \begin{bmatrix} e \\ n \\ w_s \\ q_1 \end{bmatrix} \quad (6.10)$$

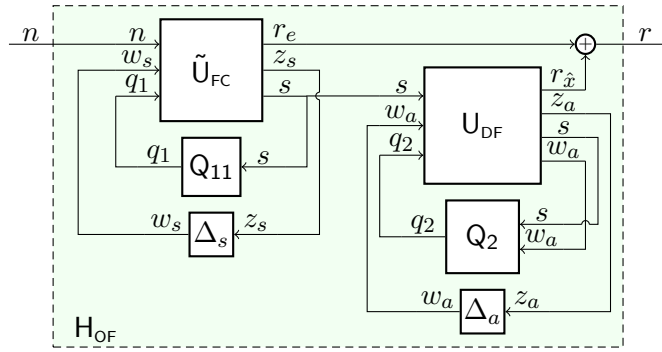


Figure 6.4: Structure of any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$ .

From here we often express  $\mathbf{H}_{\text{OF}} = F_\ell(\mathcal{T}_{\text{OF}}, \Delta) \in \mathbf{H}'_{\text{OF}}$  as the interconnection of two systems as in Figure 6.4:  $F_\ell(F_\ell(\tilde{\mathbf{U}}_{\text{FC}}, \mathbf{Q}_{11}), \Delta_s)$  and  $F_\ell(F_\ell(\mathbf{U}_{\text{DF}}, \mathbf{Q}_2), \Delta_a)$ .

Comparing (6.10) to (5.33) in Proposition 5 and similarly (6.9) to (5.18) in Proposition 3, we can see that  $F_\ell(F_\ell(\tilde{\mathbf{U}}_{\text{FC}}, \mathbf{Q}_{11}), \Delta_s)$  is an element of the subset of all MS stable networked FC closed loops and  $F_\ell(F_\ell(\mathbf{U}_{\text{DF}}, \mathbf{Q}_2), \Delta_a)$  is an element of the set of all MS stable closed loops for a networked DF plant with  $B_n = -L$  and  $D_{rn} = -L_0$ . Therefore, Corollaries 9 and 8 coupled with the implication of Lemma 10 that  $\mathbf{H}'_{\text{OF}} \neq \emptyset \Leftrightarrow \mathbf{H}_{\text{OF}} \neq \emptyset$  gives us the following key result:

**Corollary 12.** The networked OF synthesis in Problem 10 has a MS stabilizing controller if and only if  $(A, B_w, B_u, \Sigma_a)$  is MS stabilizable and  $(A, C_z, C_y, \Sigma_s)$  is MS detectable.

We define a notation which will allow us to refer explicitly to a specific parameterization.

**Notation 2.** Given  $\mathbf{H}_{\text{OF}} = \mathbf{F}_\ell(\mathbf{T}_{\text{OF}}, \Delta) \in \mathbf{H}'_{\text{OF}}$  and a realization of  $\mathbf{T}_{\text{OF}}$  parameterized by  $F$ ,  $L$ ,  $L_0$ ,  $\mathbf{Q}_{11}$ , and  $\mathbf{Q}_2$ , let

$$\mathcal{H}_{FC}(L, L_0, \mathbf{Q}_{11}, \Delta_s) = \mathbf{F}_\ell(\mathbf{F}_\ell(\tilde{\mathbf{U}}_{FC}, \mathbf{Q}_{11}), \Delta_s) \quad (6.11)$$

$$\mathcal{H}_{DF}(L, L_0, F, \mathbf{Q}_2, \Delta_a) = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{U}_{DF}, \mathbf{Q}_2), \Delta_a) \quad (6.12)$$

We will see under mild conditions, given a  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  there exist  $\bar{L}$  and  $\bar{L}_0$  for which  $\mathcal{H}_{FC}(\bar{L}, \bar{L}_0, \mathbf{Q}_{11}, \Delta_s)$  and  $\mathcal{H}_{DF}(\bar{L}, \bar{L}_0, F, \mathbf{Q}_2, \Delta_a)$  exhibit certain orthogonality properties.

Given any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$ ,  $\bar{W}_s = \lim_{k \rightarrow \infty} \mathbf{E}(w_s(k)w_s^T(k))$  is well defined according to Lemma 1(b) since  $\mathbf{H}_{\text{OF}}$  is necessarily MS stable. Given  $\bar{W}_s$ , consider the FC  $\mathcal{H}_2$  synthesis with plant (6.13) and  $\text{cov}(\bar{n}(k)) = I$ :

$$\mathbf{G}_{FC}^{\bar{W}}: \begin{bmatrix} \bar{e}^+ \\ \bar{r} \\ \bar{y} \end{bmatrix} = \left[ \begin{array}{c|cc} A & \begin{bmatrix} B_n & 0 \end{bmatrix} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_r & \begin{bmatrix} D_{rn} & 0 \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_y & \begin{bmatrix} D_n & \bar{W}_s^{\frac{1}{2}} \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} \right] \begin{bmatrix} \bar{e} \\ \bar{n}_1 \\ \bar{n}_2 \\ u_1 \\ u_2 \end{bmatrix} \quad (6.13)$$

Assume the conditions of Lemma 7 are satisfied, which guarantee the existence of the optimal solution for the above  $\mathcal{H}_2$  synthesis. Let  $X_r$  be the standard [66] Riccati equation solution and  $\bar{L}$  and  $\bar{L}_0$  be the optimal control gains for this problem, namely

$$X_r = AX_r A^T + B_n B_n^T - (AX_r C_y^T) \bar{S}^{-1} (C_y X_r A^T)$$

$$\bar{S} = (C_y X_r C_y^T + D_n D_n^T + \bar{W}_s)$$

and

$$\bar{L} = -(AX_r C_y^T) (C_y X_r C_y^T + D_n D_n^T + \bar{W}_s)^{-1} \quad (6.14a)$$

$$\bar{L}_0 = -(C_r X_r C_y^T + D_{rn} D_n^T) (C_y X_r C_y^T + D_n D_n^T + \bar{W}_s)^{-1} \quad (6.14b)$$

Since we will refer back to these quantities for a given  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  numerous times, to avoid repetitious mathematical expansions, we define some shorthand notation:

**Notation 3.** Given any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$ , let  $\mathcal{W}(\mathbf{H}_{\text{OF}}) = \bar{W}_s$

**Notation 4.** Given any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  and assuming (6.1) meets the conditions of Lemma 7, let  $\mathcal{L}^\perp(\bar{W}_s) = \bar{L}$ ,  $\mathcal{L}_0^\perp(\bar{W}_s) = \bar{L}_0$ , and  $\mathcal{S}^\perp(\bar{W}_s) = \bar{S}$ .

**Lemma 11.** Let (6.1) meet the conditions of Lemma 7. Consider any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  and let  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{\text{OF}})$ ,  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$ ,  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$ ,  $\bar{S} = \mathcal{S}^\perp(\bar{W}_s)$  and  $F$ ,  $\mathbf{Q}_{11}$ , and  $\mathbf{Q}_2$  be any parameters recovering  $\mathbf{H}_{\text{OF}}$  as the interconnection of  $\mathbf{H}_{\text{FC}} = \mathcal{H}_{\text{FC}}(\bar{L}, \bar{L}_0, \mathbf{Q}_{11}, \Delta_s)$  and  $\mathbf{H}_{\text{DF}} = \mathcal{H}_{\text{DF}}(\bar{L}, \bar{L}_0, F, \mathbf{Q}_2, \Delta_a)$  in Figure 6.4. Then we have

$$\|\mathbf{H}_{\text{OF}}\|_{\text{MSP}}^2 = \|\mathbf{H}_{\text{FC}}\|_{\text{MSP}}^2 + \|\mathbf{H}_{\text{DF}}\bar{S}^{\frac{1}{2}}\|_{\text{MSP}}^2 \quad (6.15)$$

Proof in Appendix D.

It is important to notice that the orthogonal parameterization in Lemma 11 depends on  $\bar{L}$  and  $\bar{L}_0$ , which are completely determined by (6.1),  $\Sigma_s$ , and the  $\bar{W}_s$  of the given closed loop. This means that all closed loops with the same  $\bar{W}_s$  share the same orthogonalizing gains.

Now recall that the subsystem  $\mathbf{H}_{\text{DF}} = \mathcal{H}_{\text{DF}}(\bar{L}, \bar{L}_0, F, \mathbf{Q}_2, \Delta_a) = \mathbf{F}_\ell(\mathbf{F}_\ell(\mathbf{U}_{\text{DF}}, \mathbf{Q}_2), \Delta_a)$  is an element of the set of all MS stable closed loops for a networked DF plant with  $B_n = -L$  and  $D_{rn} = -L_0$ . Assume the conditions of Lemma 3 are met, and let

$$X_{\text{DF}} = A^T X_{\text{DF}} A + C_r^T C_r - (A^T X_{\text{DF}} B_u) S_{\text{DF}}^{-1} (B_u^T X_{\text{DF}} A) \quad (6.16a)$$

$$S_{\text{DF}} = B_u^T X_{\text{DF}} B_u + D_r^T D_r + \Sigma_a \circ (B_w^T X_{\text{DF}} B_w + D_{rw}^T D_{rw}) \quad (6.16b)$$

and

$$F^* = -S_{\text{DF}}^{-1} (B_u^T X_{\text{DF}} A) \quad (6.16c)$$

$$F'_0 = S_{\text{DF}}^{-1} (B_u^T X_{\text{DF}} \bar{L} + D_r^T \bar{L}_0) \quad (6.16d)$$

Then by Corollary 7 the optimal DF closed loop for this plant is  $\mathbf{H}'_{\text{DF}} = \mathcal{H}_{\text{DF}}(\bar{L}, \bar{L}_0, F^*, [F'_0 \ 0], \Delta_a)$ .

**Notation 5.** Given any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}'_{\text{OF}}$  with  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{\text{OF}})$  and assuming (6.1) meets the conditions of Lemma 3, let  $F'_0 = \mathcal{F}(\bar{W}_s)$ .

By Proposition 4 this implies  $\|\mathbf{H}_{DF}\bar{S}^{\frac{1}{2}}\|_{MSP} \geq \|\mathbf{H}'_{DF}\bar{S}^{\frac{1}{2}}\|_{MSP}$ . Therefore, if  $\mathbf{H}'_{OF}$  is the interconnection of  $\mathbf{H}_{FC}$  and  $\mathbf{H}'_{DF}$ , then it has the same  $\bar{W}_s$ , so  $\|\mathbf{H}'_{OF}\|_{MSP}^2 = \|\mathbf{H}_{FC}\|_{MSP}^2 + \|\mathbf{H}'_{DF}\bar{S}^{\frac{1}{2}}\|_{MSP}^2 \Rightarrow \|\mathbf{H}'_{OF}\|_{MSP} \leq \|\mathbf{H}_{OF}\|_{MSP}$ . Based on this we further refine the set of closed loops under consideration.

**Definition 11.** Let  $\mathbf{H}''_{OF} \subset \mathbf{H}'_{OF}$  be the set of all  $\mathbf{H}_{OF} \in \mathbf{H}'_{OF}$  which can be parameterized as the interconnection of  $\mathbf{H}_{FC} = \mathcal{H}_{FC}(\bar{L}, \bar{L}_0, \mathbf{Q}_{11}, \Delta_s)$  and  $\mathbf{H}_{DF} = \mathcal{H}_{DF}(\bar{L}, \bar{L}_0, F^*, [F'_0 \ 0], \Delta_a)$  where  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{OF})$ ,  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$ ,  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$ , and  $F'_0 = \mathcal{F}(\bar{W}_s)$ .

**Remark 5.** Recalling (6.16) we note a point that will be crucial later on:  $X_{DF}$  and  $F^*$  are completely determined by the plant (6.1) and  $\Sigma_a$ , i.e. these quantities are invariant among all  $\mathbf{H}_{OF} \in \mathbf{H}''_{OF}$ .

The remaining question is which  $\mathbf{H}_{OF} \in \mathbf{H}''_{OF}$  has the optimal MS performance? To enable us to answer this question we next show that the MS performance of any  $\mathbf{H}_{OF} \in \mathbf{H}''_{OF}$  is equal to the optimal  $\mathcal{H}_2$  performance of a problem which is parameterized by a noise covariance  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{OF})$ .

**Corollary 13.** Given any  $\mathbf{H}_{OF} \in \mathbf{H}''_{OF}$ ,  $\|\mathbf{H}_{OF}\|_{MSP}$  is equal to the optimal  $\mathcal{H}_2$  performance for a standard synthesis problem with generalized plant (6.17) where  $\bar{y}(k)$  and  $\bar{u}(k)$  are the measurement and control input at time  $k$ ,  $\text{cov}(\bar{n}(k)) = I$ ,  $\bar{V} = \Sigma_a \circ (B_w^T X_{DF} B_w + D_{rw}^T D_{rw})$ , and  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{OF})$ .

$$\mathbf{G}_{OF}^{\bar{W}_s} = \left[ \begin{array}{c|cc} \bar{x}_G^+ & A & \begin{bmatrix} B_n & 0 \end{bmatrix} & B_u \\ \hline \bar{r}_1 & C_r & \begin{bmatrix} D_{rn} & 0 \end{bmatrix} & D_r \\ \bar{r}_2 & 0 & \begin{bmatrix} 0 & 0 \end{bmatrix} & \bar{V}^{\frac{1}{2}} \\ \bar{y} & C_y & \begin{bmatrix} D_n & \bar{W}_s^{\frac{1}{2}} \end{bmatrix} & 0 \end{array} \right] \begin{bmatrix} \bar{x}_G \\ \bar{n}_1 \\ \bar{n}_2 \\ \bar{u} \end{bmatrix} \quad (6.17)$$

Proof in Appendix D.

Notice  $\bar{V}$  in (6.17) is calculated using  $X_{DF}$  from (6.16) and therefore, recalling Remark 5, it is invariant among all  $\mathbf{H}_{OF} \in \mathbf{H}''_{OF}$ . The only part of the plant (6.17) that changes from one element of  $\mathbf{H}''_{OF}$  to the next is  $\bar{W}_s$ , i.e. an exogenous noise covariance from the  $\mathcal{H}_2$  perspective.

As may be intuitive, we will see next that the optimal networked FC solution provides the minimal  $\bar{W}_s$  among those generated in  $\mathbf{H}_{\text{OF}}''$ . Therefore from Corollary 13, it follows that such solution will lead to the smallest performance.

## 6.2 Optimal Networked Output Feedback

**Definition 12.** *Given the networked OF synthesis in Problem 10, let (6.1) meet the conditions of Lemmas 3 and 7 be met, and let*

$$X_{FC} = AX_{FC}A^T + B_n B_n^T - (AX_{FC}C_y^T)S_{FC}^{-1}(C_y X_{FC}A^T) \quad (6.18a)$$

$$R_{FC} = C_r X_{FC}C_r^T + D_{rn}D_{rn}^T - (C_r X_{FC}C_y^T + D_{rn}D_n^T)S_{FC}^{-1}(C_y X_{FC}C_r^T + D_n D_{rn}^T) \quad (6.18b)$$

$$S_{FC} = C_y X_{FC}C_y^T + D_n D_n^T + \Sigma_s \circ (C_z X_{FC}C_z^T + D_{zn}D_{zn}^T) \quad (6.18c)$$

and

$$L^* = -(AX_{FC}C_y^T)S_{FC}^{-1} \quad (6.18d)$$

$$L_0^* = -(C_r X_{FC}C_y^T + D_{rn}D_n^T)S_{FC}^{-1} \quad (6.18e)$$

$$X_{DF} = A^T X_{DF}A + C_r^T C_r - (A^T X_{DF}B_u)S_{DF}^{-1}(B_u^T X_{DF}A) \quad (6.19a)$$

$$R_{DF} = L^{*T} X_{DF}L^* + L_0^{*T} L_0^* - (L^{*T} X_{DF}B_u + L_0^{*T} D_r)S_{DF}^{-1}(B_u^T X_{DF}L^* + D_r^T L_0^*) \quad (6.19b)$$

$$S_{DF} = B_u^T X_{DF}B_u + D_r^T D_r + \Sigma_a \circ (B_w^T X_{DF}B_w + D_{rw}^T D_{rw}) \quad (6.19c)$$

and

$$F^* = -S_{DF}^{-1}(B_u^T X_{DF}A) \quad (6.19d)$$

$$F_0^* = S_{DF}^{-1}(B_u^T X_{DF}L^* + D_r^T L_0^*) \quad (6.19e)$$

Now we give the main synthesis result in this work which allows us to solve Problem 10 optimally as a sequence of optimal networked FC and DF design steps. A dual characterization exists in terms of a sequence of optimal networked FI and OE design steps, which we omit. We state the result in terms of the Riccati-like quantities of Lemmas 3 and 7 for sake of compactness

and connection to the development and steps in the proofs. However, recall from the discussion in Section 5.1 that the optimal control gains are also obtainable from the limit points of the LMIs in Lemmas 2 and 6 which also establish MS stabilizability and detectability.

**Theorem 2.** *Given Problem 10, let the conditions of Lemmas 3 and 7 be met, then*

$$(\nu_{OF}^*)^2 = \|F_\ell(F_\ell(G_{OF}, K_{OF}^*), \Delta)\|_{MSP}^2 = \text{tr}(R_{FC}) + \text{tr}(S_{FC}R_{DF})$$

where  $K_{OF}^*$  has realization

$$K_{OF}^* : \begin{bmatrix} x_K^+ \\ u_s \\ u_a \end{bmatrix} = \left[ \begin{array}{c|cc} A + B_u F^* + L^* C_y - B_u F_0^* C_y & B_u F_0^* - L^* & B_w \\ \hline -C_z & 0 & 0 \\ F^* - F_0^* C_y & F_0^* & 0 \end{array} \right] \begin{bmatrix} x_K \\ y_s \\ y_a \end{bmatrix} \quad (6.20)$$

Moreover,  $\text{tr}(R_{FC}) = \|H_{FC}^*\|_{MSP}^2$  where  $H_{FC}^* = F_\ell(T_{FC}^*, \Delta_s)$  with

$$T_{FC}^* : \begin{bmatrix} x_{FC}^+ \\ r_{FC} \\ z_s \end{bmatrix} = \left[ \begin{array}{c|cc} A + L^* C_y & B_n + L^* D_n & L^* \\ \hline C_r + L_0^* C_y & D_{rn} + L_0^* D_n & L_0^* \\ C_z & D_{zn} & 0 \end{array} \right] \begin{bmatrix} x_{FC} \\ n \\ w_s \end{bmatrix} \quad (6.21)$$

and  $\text{tr}(S_{FC}R_{DF}) = \|H_{DF}^* S_{FC}^{\frac{1}{2}}\|_{MSP}^2$  where  $H_{DF}^* = F_\ell(T_{DF}^*, \Delta_s)$  with

$$T_{DF}^* : \begin{bmatrix} x_{DF}^+ \\ r_{DF} \\ z_a \end{bmatrix} = \left[ \begin{array}{c|cc} A + B_u F^* & -L^* + B_u F_0^* & B_w \\ \hline C_r + D_r F^* & -L_0^* + D_r F_0^* & D_{rw} \\ F^* & F_0^* & 0 \end{array} \right] \begin{bmatrix} x_{DF} \\ s \\ w_a \end{bmatrix} \quad (6.22)$$

Proof in Appendix D

### 6.3 Networked Output Feedback Implementation

Recall the networked control setup in Figures 4.1 and 4.2, the plant P in (4.1), and the controller  $\hat{K}$  in (4.4). It can be verified the optimal controller (6.20) in Theorem 2 is equivalent to

$$\hat{K} : \begin{bmatrix} x_K^+ \\ u_1 \\ u_2 \end{bmatrix} = \left[ \begin{array}{c|cc} A & -L^* M_s^{-1} & AB_u \\ \hline -C_y & 0 & -C_y B_u \\ M_a^{-1} F^* & M_a^{-1} F_0^* M_s^{-1} & M_a^{-1} F^* B_u \end{array} \right] \begin{bmatrix} x_K \\ y_1 \\ \hat{y}_2 \end{bmatrix} \quad (6.23)$$

This can be implemented as by having  $\mathcal{K}$  store  $u(k-1)$  and use  $\alpha_a(k)$  to form  $\hat{y}_2(k) = \Xi_a(k-1)u(k-1)$ , and similarly using  $u_1(k)$  and  $\alpha_s(k)$  to form  $y_1(k) = y(k) + \Xi_s(k)u_1(k)$ .

## CHAPTER 7. NUMERICAL EXAMPLES

### 7.1 Network Limitations and Trade-off Analysis

In the following examples we concentrate on multiple-input systems which are stabilizable from each input and study the trade-offs between the uncertainties of the channels. We consider two such systems, and for each compare our numerically calculated boundary to a closed-form result for such systems found in [31, 32].

**Example 1.** Consider a plant with the following  $A$ ,  $B_w$  and  $B_u$ .

$$A = \begin{bmatrix} 1.1006 & -1.4916 & 2.3505 \\ 1.5442 & -0.7423 & -0.6156 \\ 0.0859 & -1.0616 & 0.7481 \end{bmatrix} \quad B_w = \begin{bmatrix} -0.1924 & -1.4023 \\ 0.8886 & -1.4224 \\ -0.7648 & 0.4882 \end{bmatrix} \quad B_u = \begin{bmatrix} -0.1924 & -1.4023 \\ 0.8886 & -1.4224 \\ -0.7648 & 0.4882 \end{bmatrix}$$

This example has one real and two complex conjugate unstable eigenvalues at  $-1.0362$  and  $1.0713 \pm 1.404i$  respectively. Figure 7.1 depicts the limitation curve calculated using Corollary 6 for this system in red, as well as the closed-form necessity condition (7.2) found in [31, 32] in blue.

**Example 2.** Consider a plant with the following  $A$ ,  $B_w$  and  $B_u$ .

$$A = \begin{bmatrix} 0.3999 & -0.1768 \\ -0.9300 & -2.1321 \end{bmatrix} \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This example has a single unstable eigenvalue at  $-2.1955$ . Figure 7.2 shows our boundary from Corollary 6 in red and the necessary condition from the literature in blue for this example.

Note that in both figures, the region where the system is mean-square stabilizable is above the calculated boundary. We also note that the limitations curves for the two examples are different from each other. This is an indication that the MIMO limitations are more complex than in the single input case.

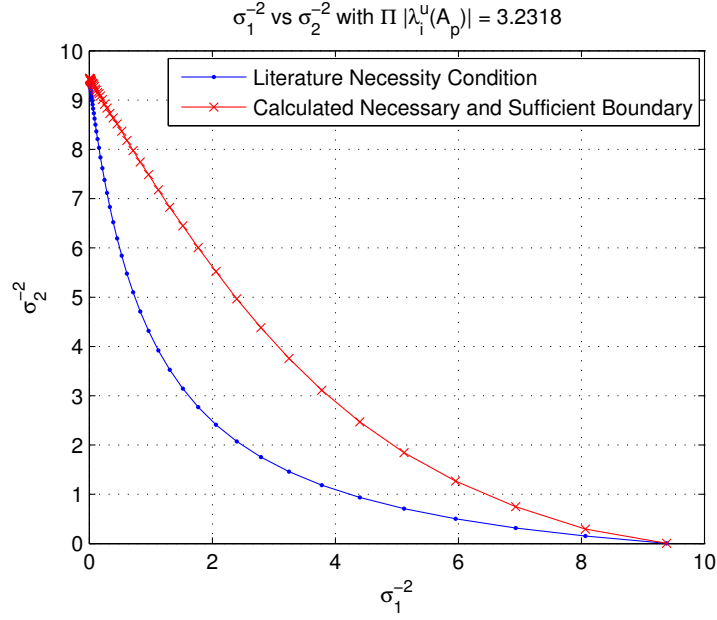


Figure 7.1: Comparison of MS Stabilizability Limitations for Example 1.

We also see from the above examples that the actual limitation curve can be quite different from the necessary curve. Moreover, we see that the necessary condition becomes sufficient on the axes. These points correspond to using only one of the channels, and in this case the limitation reduces to the known single-channel result in [23], namely

$$C = \frac{1}{2} \log(1 + \sigma^{-2}) > \sum_i \log |\lambda_i^{(u)}(A_p)| \quad (7.1)$$

where  $\lambda_i^{(u)}(A_p)$  is the  $i^{\text{th}}$  unstable eigenvalue of  $A_p$ . Corresponding results for those situations involving multiple channels have been limited as they have applied the single-input limitation to multiple-input systems. This is done by breaking down the stabilization problem into independent sub-problems each involving a single input and an associated subset of unstable eigenvalues.

In particular, the works [31, 32] have provided the necessary condition for mean-square stabilizability shown in our examples. The condition for a  $p$ -channel system given in these works is

$$C = \sum_{i=1}^p \frac{1}{2} \log(1 + \sigma_i^{-2}) > \sum_i \log |\lambda_i^{(u)}(A_p)| \quad (7.2)$$



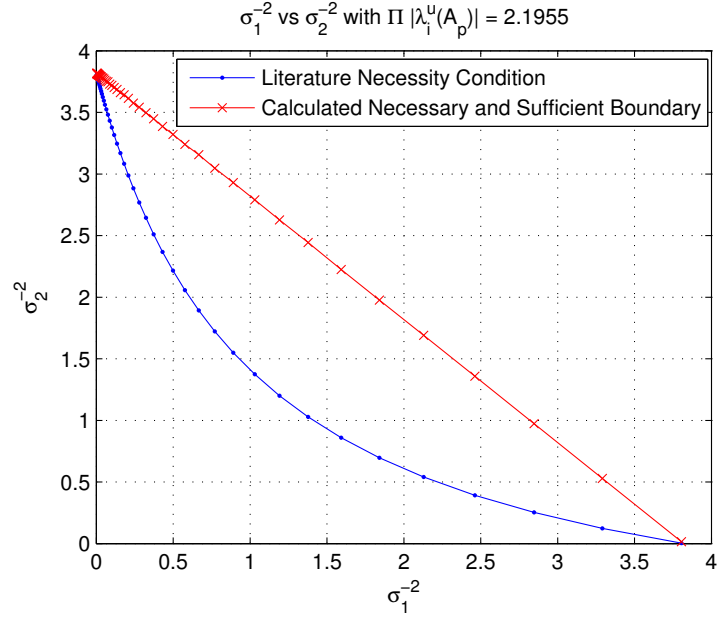


Figure 7.2: Comparison of MS Stabilizability Limitations for Example 2.

This condition can become sufficient for some  $\sigma_i^2$ , namely when the  $\sigma_i^2$  can be freely allocated subject only to the overall constraint  $\sum_{i=1}^p \frac{1}{2} \log(1 + \sigma_i^{-2}) = C$  for some total  $C$ . Generally speaking however, (7.2) does not determine the sufficiency of any particular given set of  $\sigma_i^2$ . Moreover, as we have seen in our examples the allocation that makes condition (11) sufficient is to just use one of the channels when multiple channels can stabilize the same eigenvalues.

In contrast, our results provide the necessary and sufficient boundary curve when all channels are used for stabilization with various  $\sigma_i^2$ , as well as the feasible region of  $\sigma_i^2$  sufficient for mean-square stabilizability of the closed loop.

## 7.2 Westland Lynx Military Helicopter

We consider the linearized model of the Westland Lynx multipurpose British military helicopter from [68]. This system has four inputs: main rotor collective pitch, longitudinal and lateral cyclic pitch, and tail rotor collective pitch. *Roughly speaking*, these inputs control heave motion, longitudinal and lateral motion, and yaw respectively. It is assumed six measurements are available: heave velocity, pitch attitude, roll attitude, heading rate, pitch rate, and roll rate.

The matrices  $A \in \mathbf{R}^{n \times n}$ ,  $B_u \in \mathbf{R}^{n \times m}$ , and  $C_y \in \mathbf{R}^{l \times n}$  are obtained by discretization of the continuous time state space model with a period of 0.01 s. For MS performance synthesis, we partition  $r(k) = \begin{bmatrix} r_x(k) \\ r_u(k) \end{bmatrix}$  and  $n(k) = \begin{bmatrix} n_x(k) \\ n_y(k) \end{bmatrix}$  and choose a simple<sup>1</sup> set of performance and noise weights:  $C_r = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$   $D_{rn} = 0$   $D_r = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$   $B_n = \begin{bmatrix} B_u & 0 \end{bmatrix}$   $D_n = \begin{bmatrix} 0 & I_l \end{bmatrix}$  which gives

$$P : \begin{bmatrix} x_P^+ \\ r_x \\ r_u \\ y_P \end{bmatrix} = \left[ \begin{array}{c|cc} A & \begin{bmatrix} B_u & 0 \end{bmatrix} & B_u \\ \hline \begin{bmatrix} I_n \\ 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\ \hline C_y & \begin{bmatrix} 0 & I_l \end{bmatrix} & 0 \end{array} \right] \begin{bmatrix} x_P \\ n_x \\ n_y \\ u_P \end{bmatrix}$$

**Example 3** (Actuator vs. Sensor Dropouts). *In Figure 7.3 we show the optimal MS performance as a function of sensor and actuator dropout rate assuming that dropouts on all ten channels are independent. We see a noticeable difference in overall sensitivity to dropout between sensor and actuator links. The MS performance surface approaches infinity much more sharply on the actuator axis than the sensor axis when nearing the boundary of MS stabilizability, but is less affected for lower dropout percentages.*

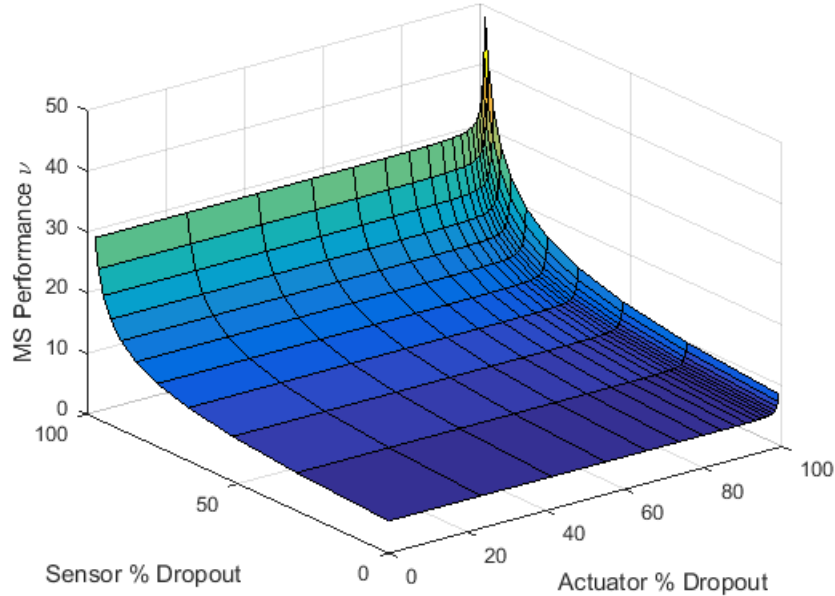


Figure 7.3: MS Performance vs. Dropout Rates

<sup>1</sup>Note that we choose these weights for simple illustrative purposes.

**Example 4** (Sensor and Actuator Group Dropout). In Figure 7.4 we show the optimal MS performance vs. dropout rate tradeoff curves for sensor and actuator signals. We partition the available sensor readings into groups: heave velocity (e.g. GPS/accelerometer), pitch and roll attitude (e.g. inclinometer), heading rate (e.g. magnetometer), and body pitch and roll rates (e.g. gyroscope). Similarly for the actuators: main rotor collective, longitudinal and lateral cyclic, and tail rotor collective.

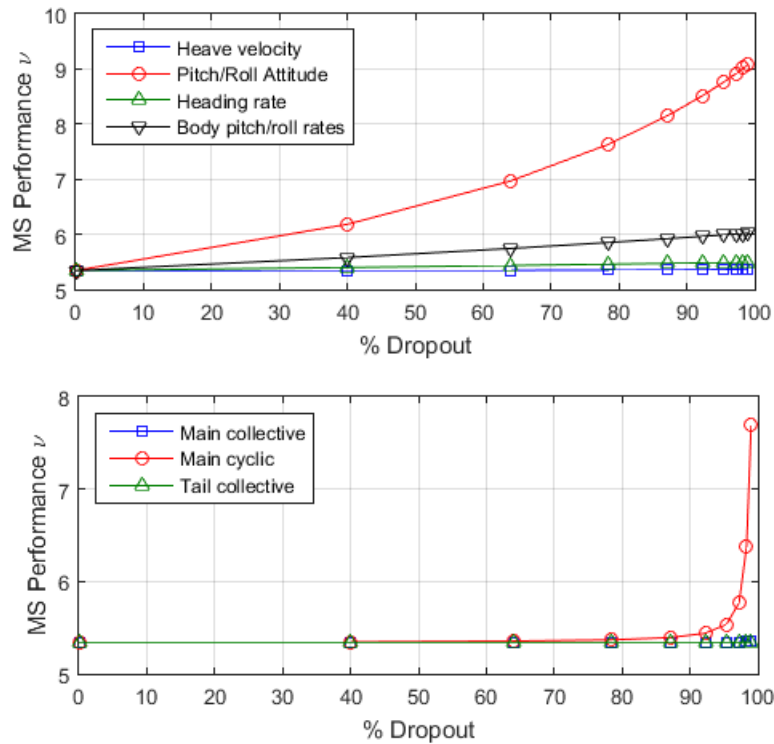


Figure 7.4: MS Performance vs. Combined Dropout Rates

## APPENDIX A. SOME GENERAL AUXILIARY THEORY

Here we provide some results which will be useful in various proofs in subsequent appendices. We begin by defining a generic linear operator  $\Gamma$  which can be used to describe numerous quantities throughout this work as special cases. We then address the convergence (stability) properties of this operator and existence and uniqueness of solutions to matrix equations involving it.

**Definition 13.** Let  $\Gamma : \mathbf{R}^{m \times n} \mapsto \mathbf{R}^{m \times n}$  be a linear operator  $X = \sum_i^M A_i X B_i$  for  $0 < M \in \mathbf{N}$

**Proposition 12.** Let  $\Gamma$  be as given in Definition 13 and  $X(k+1) = \Gamma(X(k))$ . Then  $X(k) \rightarrow 0$  as  $k \rightarrow \infty$  for any  $X(0)$  if and only if  $\rho(\Gamma) < 1$ .

*Proof of Proposition 12.* We can vectorize the recursion  $X(k+1) = \Gamma(X(k))$  using Kronecker products as  $\text{vec}(X(k+1)) = \sum_i^M (B_i^T \otimes A_i) \text{vec}(X(k))$  where  $\rho(\Gamma) < 1$  implies the maximum eigenvalue of the matrix  $\sum_i^M (B_i^T \otimes A_i)$  has magnitude less than 1. Therefore this follows from standard linear systems theory.  $\square$

**Proposition 13.** If  $\rho(\Gamma) < 1$ , then the unique solution to  $X = \Gamma(X)$  is  $X = 0$ .

*Proof of Proposition 13.* It is trivial to verify  $X = 0$  is a solution. To see that it is unique, assume  $Z = \Gamma(Z)$  where  $Z \neq 0$ . Then  $\Gamma$  has an eigenvalue at 1, which contradicts  $\rho(\Gamma) < 1$ .  $\square$

**Proposition 14.** If  $\rho(\Gamma) < 1$  and  $Y \in \mathbf{R}^{m \times n}$ , then  $X = Y + \sum_{k=1}^{\infty} \Gamma^k(Y)$  is well defined and is the unique solution  $X = \Gamma(X) + Y$ .

*Proof of Proposition 14.* Since  $\Gamma$  is linear and  $\rho(\Gamma) < 1$ ,  $\Gamma^k(Y) \rightarrow 0$  exponentially and therefore  $X = Y + \sum_{k=1}^{\infty} \Gamma^k(Y)$  is well defined. We can then verify  $X = \Gamma(X) + Y$  by inspection. To show the uniqueness, assume  $Z \neq X$  is such that  $Z = \Gamma(Z) + Y$ . Then  $X - Z = \Gamma(X - Z) \neq 0$  which contradicts Proposition 13.  $\square$

Now define a special case of the more generic linear operator we just considered, namely the symmetric version. We will see that the stability of this operator can be related to the feasibility of a linear matrix inequality. This concept, which follows from [60], forms the fundamental basis for a large portion of the analysis and synthesis theory in this work.

**Definition 14.** Let  $\Lambda : \mathbf{S}^n \mapsto \mathbf{S}$  be a linear operator  $X = \sum_i^M A_i X A_i^T$  for  $0 < M \in \mathbf{N}$

**Proposition 15.** Let  $\Lambda$  be as given in Definition 14 and  $X(k+1) = \Lambda(X(k))$ . Then  $X(k) \rightarrow 0$  as  $k \rightarrow \infty$  for any  $X(0) \succeq 0$  if and only if there exists a  $Q \succ 0$  such that  $Q \succ \Lambda(Q)$ .

Proposition 15 follows directly from Chapter 9 of [60].

## APPENDIX B. PROOFS AND AUXILIARY THEORY FOR CHAPTER 3

*Proof of Lemma 1.* (a) Assumption 5 guarantees  $z(k)$  and  $Z(k)$  are well defined  $\forall k \geq 0$  by ensuring  $(I - \Delta(k)\bar{D}_{zw})^{-1}$  exists  $\forall k \geq 0$ . Moreover it keeps  $\delta_i(k)$  and the elements of  $z(k)$  multiplied by  $\delta_i(k)$  in  $\Delta(k)z(k)$  independent. This implies

$$W(k) = E(\Delta(k)z(k)z^T(k)\Delta(k)) = \Sigma \circ Z(k) \quad (\text{B.1})$$

Since  $E(w(k)x^T(k)) = 0$  and  $E(w(k)n^T(k)) = 0 \forall k \geq 0$  we have  $Z(k) = E(z(k)z^T(k))$  is

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \bar{D}_{zw} W(k) \bar{D}_{zw}^T \quad (\text{B.2})$$

Note (B.1) and (B.2) are (3.1d) and (3.1c) respectively. Similarly calculating  $R(k)$  and  $X(k+1)$  we get

$$R(k) = \bar{C}_r X(k) \bar{C}_r^T + \bar{D}_{rn} N \bar{D}_{rn}^T + \bar{D}_{rw} W(k) \bar{D}_{rw}^T \quad (\text{B.3})$$

$$X(k+1) = \bar{A} X(k) \bar{A}^T + \bar{B}_n N \bar{B}_n^T + \bar{B}_w W(k) \bar{B}_w^T \quad (\text{B.4})$$

which are (3.1b) and (3.1a) respectively.

(b) This follows from part (a) and Definitions 1 and 2.

(c) This follows from part (a) and Definitions 1 and 2. □

*Proof of Corollary 2.* Consider two realizations for  $\mathbb{T}$  formed with matrices

$$(\hat{A}, \hat{B}_n, \hat{B}_w, \hat{C}_r, \hat{D}_{rn}, \hat{D}_{rw}, \hat{C}_z, \hat{D}_{zn}, \hat{D}_{zw}) \text{ and } (\tilde{A}, \tilde{B}_n, \tilde{B}_w, \tilde{C}_r, \tilde{D}_{rn}, \tilde{D}_{rw}, \tilde{C}_z, \tilde{D}_{zn}, \tilde{D}_{zw})$$

respectively. Because these matrices realize the same transfer function, they produce an identical impulse response. We therefore have that  $\forall k \in \mathbf{N}$

$$\begin{bmatrix} \hat{D}_{rn} & \hat{D}_{rw} \\ \hat{D}_{zn} & \hat{D}_{zw} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{rn} & \tilde{D}_{rw} \\ \tilde{D}_{zn} & \tilde{D}_{zw} \end{bmatrix} \text{ and } \begin{bmatrix} \hat{C}_r \\ \hat{C}_z \end{bmatrix} \hat{A}^k \begin{bmatrix} \hat{B}_n & \hat{B}_w \end{bmatrix} = \begin{bmatrix} \tilde{C}_r \\ \tilde{C}_z \end{bmatrix} \tilde{A}^k \begin{bmatrix} \tilde{B}_n & \tilde{B}_w \end{bmatrix} \quad (\text{B.5})$$

Applying Lemma 1 to  $\mathbf{H} = \mathbf{F}_\ell(\mathbf{T}, \Delta)$  using the first realization gives us that  $\forall k \geq 0$

$$\hat{X}(k+1) = \hat{A}\hat{X}(k)\hat{A}^T + \hat{B}_n N \hat{B}_n^T + \hat{B}_w \hat{W}(k) \hat{B}_w^T \quad (\text{B.6a})$$

$$\hat{R}(k) = \hat{C}_r \hat{X}(k) \hat{C}_r^T + \hat{D}_{rn} N \hat{D}_{rn}^T + \hat{D}_{rw} \hat{W}(k) \hat{D}_{rw}^T \quad (\text{B.6b})$$

$$\hat{Z}(k) = \hat{C}_z \hat{X}(k) \hat{C}_z^T + \hat{D}_{zn} N \hat{D}_{zn}^T + \hat{D}_{zw} \hat{W}(k) \hat{D}_{zw}^T \quad (\text{B.6c})$$

$$\hat{W}(k) = \Sigma \circ \hat{Z} \quad (\text{B.6d})$$

Similarly using the second realization we have that  $\forall k \geq 0$

$$\tilde{X}(k+1) = \tilde{A}\tilde{X}(k)\tilde{A}^T + \tilde{B}_n N \tilde{B}_n^T + \tilde{B}_w \tilde{W}(k) \tilde{B}_w^T \quad (\text{B.7a})$$

$$\tilde{R}(k) = \tilde{C}_r \tilde{X}(k) \tilde{C}_r^T + \tilde{D}_{rn} N \tilde{D}_{rn}^T + \tilde{D}_{rw} \tilde{W}(k) \tilde{D}_{rw}^T \quad (\text{B.7b})$$

$$\tilde{Z}(k) = \tilde{C}_z \tilde{X}(k) \tilde{C}_z^T + \tilde{D}_{zn} N \tilde{D}_{zn}^T + \tilde{D}_{zw} \tilde{W}(k) \tilde{D}_{zw}^T \quad (\text{B.7c})$$

$$\tilde{W}(k) = \Sigma \circ \tilde{Z} \quad (\text{B.7d})$$

Now if  $\Sigma = \text{diag}(\sigma_1^2 \mathbf{1}_{\eta_1 \times \eta_1}, \dots, \sigma_q^2 \mathbf{1}_{\eta_q \times \eta_q}) \in \mathbf{B}_+^\eta$  and we define the set of matrices

$$\{E_1 = \text{diag}(\sigma_1 I_{\eta_1}, 0, \dots), E_2 = \text{diag}(0, \sigma_2 I_{\eta_2}, 0, \dots), \dots, E_q = \text{diag}(0, \dots, \sigma_q I_{\eta_q})\}$$

then we can express  $\Sigma \circ \hat{Z}(k)$  and  $\Sigma \circ \tilde{Z}(k)$  as  $\sum_i E_i \hat{Z}(k) E_i$  and  $\sum_i E_i \tilde{Z}(k) E_i$  respectively.

Additionally, it is known that if  $AXB = C$  then  $\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X) = \text{vec}(C)$ , where  $\text{vec}$  is the vectorization operator and  $\otimes$  represents the Kronecker product. Using these facts, (B.6a)-(B.6d) and (B.7a)-(B.7d) are equivalent to standard LTI feedback interconnections:  $\mathbf{F}_\ell(\hat{\mathbf{T}}, \underline{E})$  and  $\mathbf{F}_\ell(\tilde{\mathbf{T}}, \underline{E})$  respectively where

$$\hat{\mathbf{T}} : \begin{bmatrix} \hat{x}^+ \\ \hat{r} \\ \hat{z} \end{bmatrix} \left[ \begin{array}{c|cc} \hat{A} & \hat{B}_n & \hat{B}_w \\ \hline \hat{C}_r & \hat{D}_{rn} & \hat{D}_{rw} \\ \hat{C}_z & \hat{D}_{zn} & \hat{D}_{zw} \end{array} \right] \begin{bmatrix} \hat{x} \\ \hat{n} \\ \hat{w} \end{bmatrix} \quad (\text{B.8})$$

$$\tilde{\mathbf{T}} : \begin{bmatrix} \tilde{x}^+ \\ \tilde{r} \\ \tilde{z} \end{bmatrix} \left[ \begin{array}{c|cc} \tilde{A} & \tilde{B}_n & \tilde{B}_w \\ \hline \tilde{C}_r & \tilde{D}_{rn} & \tilde{D}_{rw} \\ \tilde{C}_z & \tilde{D}_{zn} & \tilde{D}_{zw} \end{array} \right] \begin{bmatrix} \tilde{x} \\ \tilde{n} \\ \tilde{w} \end{bmatrix} \quad (\text{B.9})$$

$$\hat{x}(k) = \text{vec}(\hat{X}(k)), \hat{r}(k) = \text{vec}(\hat{R}(k)), \hat{z}(k) = \text{vec}(\hat{Z}(k)), \hat{n}(k) = \text{vec}(N), \hat{w}(k) = \text{vec}(\hat{W}(k)),$$

$$\begin{aligned}
\hat{\underline{A}} &= \hat{A} \otimes \hat{A} & \hat{\underline{B}}_n &= \hat{B}_n \otimes \hat{B}_n & \hat{\underline{B}}_w &= \hat{B}_w \otimes \hat{B}_w \\
\hat{\underline{C}}_r &= \hat{C}_r \otimes \hat{C}_r & \hat{\underline{D}}_{rn} &= \hat{D}_{rn} \otimes \hat{D}_{rn} & \hat{\underline{D}}_{rw} &= \hat{D}_{rw} \otimes \hat{D}_{rw} \\
\hat{\underline{C}}_z &= \hat{C}_z \otimes \hat{C}_z & \hat{\underline{D}}_{zn} &= \hat{D}_{zn} \otimes \hat{D}_{zn} & \hat{\underline{D}}_{zw} &= \hat{D}_{zw} \otimes \hat{D}_{zw}
\end{aligned}$$

$$\tilde{x}(k) = \text{vec}(\tilde{X}(k)), \tilde{r}(k) = \text{vec}(\tilde{R}(k)), \tilde{z}(k) = \text{vec}(\tilde{Z}(k)), \tilde{n}(k) = \text{vec}(N), \tilde{w}(k) = \text{vec}(\tilde{W}(k)),$$

$$\begin{aligned}
\tilde{\underline{A}} &= \tilde{A} \otimes \tilde{A} & \tilde{\underline{B}}_n &= \tilde{B}_n \otimes \tilde{B}_n & \tilde{\underline{B}}_w &= \tilde{B}_w \otimes \tilde{B}_w \\
\tilde{\underline{C}}_r &= \tilde{C}_r \otimes \tilde{C}_r & \tilde{\underline{D}}_{rn} &= \tilde{D}_{rn} \otimes \tilde{D}_{rn} & \tilde{\underline{D}}_{rw} &= \tilde{D}_{rw} \otimes \tilde{D}_{rw} \\
\tilde{\underline{C}}_z &= \tilde{C}_z \otimes \tilde{C}_z & \tilde{\underline{D}}_{zn} &= \tilde{D}_{zn} \otimes \tilde{D}_{zn} & \tilde{\underline{D}}_{zw} &= \tilde{D}_{zw} \otimes \tilde{D}_{zw}
\end{aligned}$$

and  $\underline{E} = \sum_{i=1}^q E_i \otimes E_i$ .

Recalling (B.5) and the fact that  $(A \otimes B)(C \otimes D) = AC \otimes BD$  one can easily verify  $\hat{\mathbb{T}} = \tilde{\mathbb{T}}$  which due to standard properties of LFT implies  $F_\ell(\hat{\mathbb{T}}, \underline{E}) = F_\ell(\tilde{\mathbb{T}}, \underline{E})$ . Moreover,  $\hat{n}(k) = \tilde{n}(k) \forall k \geq 0$ . Therefore, if we start from  $\hat{x}(0) = 0$  and  $\tilde{x}(0) = 0$  then we necessarily have that  $\hat{r}(k) = \tilde{r}(k)$ ,  $\hat{z}(k) = \tilde{z}(k)$ , and  $\hat{w}(k) = \tilde{w}(k) \forall k \geq 0$ . Finally, we know that the steady state limits will be invariant with respect to initial conditions, and we conclude  $\lim_{k \rightarrow \infty} \hat{R}(k) = \lim_{k \rightarrow \infty} \tilde{R}(k)$ ,  $\lim_{k \rightarrow \infty} \hat{Z}(k) = \lim_{k \rightarrow \infty} \tilde{Z}(k)$ , and  $\lim_{k \rightarrow \infty} \hat{W}(k) = \lim_{k \rightarrow \infty} \tilde{W}(k)$ .  $\square$

*Proof of Corollary 3.* Consider the realizations for  $\mathbb{T}$  and  $\mathbb{T}^T$  formed with matrices

$$(\bar{A}, \bar{B}_n, \bar{B}_w, \bar{C}_r, \bar{D}_{rn}, \bar{D}_{rw}, \bar{C}_z, \bar{D}_{zn}, \bar{D}_{zw}) \text{ and } (\bar{A}^T, \bar{C}_r^T, \bar{C}_z^T, \bar{B}_n^T, \bar{D}_{rn}^T, \bar{D}_{zn}^T, \bar{B}_w^T, \bar{D}_{rw}^T, \bar{D}_{zw}^T)$$

respectively. We are interested in  $\|\mathbb{H}\|_{MSP}$  and  $\|\mathbb{H}^T\|_{MSP}$ . Therefore we apply Lemma 1 to each system assuming  $\text{cov}(n) = I$ . For  $\mathbb{H} = F_\ell(\mathbb{T}, \Delta)$  this gives us that  $\forall k \geq 0$

$$\hat{X}(k+1) = \bar{A}\hat{X}(k)\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_w\hat{W}(k)\bar{B}_w^T \quad (\text{B.10a})$$

$$\hat{R}(k) = \bar{C}_r\hat{X}(k)\bar{C}_r^T + \bar{D}_{rn}\bar{D}_{rn}^T + \bar{D}_{rw}\hat{W}(k)\bar{D}_{rw}^T \quad (\text{B.10b})$$

$$\hat{Z}(k) = \bar{C}_z\hat{X}(k)\bar{C}_z^T + \bar{D}_{zn}\bar{D}_{zn}^T + \bar{D}_{zw}\hat{W}(k)\bar{D}_{zw}^T \quad (\text{B.10c})$$

$$\hat{W}(k) = \Sigma \circ \hat{Z} \quad (\text{B.10d})$$



Similarly, for  $\mathbf{H}^T = \mathbf{F}_\ell(\mathbf{T}^T, \Delta)$  we have that  $\forall k \geq 0$

$$\tilde{X}(k+1) = \bar{A}^T \tilde{X}(k) \bar{A} + \bar{C}_r^T \bar{C}_r + \bar{C}_z^T \tilde{W}(k) \bar{C}_z \quad (\text{B.11a})$$

$$\tilde{R}(k) = \bar{B}_n^T \tilde{X}(k) \bar{N}_n + \bar{D}_{rn}^T \bar{D}_{rn} + \bar{D}_{zn}^T \tilde{W}(k) \bar{D}_{zn} \quad (\text{B.11b})$$

$$\tilde{Z}(k) = \bar{B}_w^T \tilde{X}(k) \bar{B}_w + \bar{D}_{rw}^T \bar{D}_{rw} + \bar{D}_{zw}^T \tilde{W}(k) \bar{D}_{zw} \quad (\text{B.11c})$$

$$\tilde{W}(k) = \Sigma \circ \tilde{Z} \quad (\text{B.11d})$$

Following analogous steps as in the proof of Corollary 2 we can form an equivalent LTI feedback interconnection for both (B.10a)-(B.10d) and (B.11a)-(B.11d) using vectorization and Kronecker products, e.g.  $\mathbf{F}_\ell(\hat{\mathbf{T}}, \underline{E})$  and  $\mathbf{F}_\ell(\tilde{\mathbf{T}}, \underline{E})$ . In this case using the property of Kronecker product that  $(A \otimes B)^T = A^T \otimes B^T$ , one can easily verify that  $\hat{\mathbf{T}} = \tilde{\mathbf{T}}^T$ . Since  $\underline{E} = \underline{E}^T$  this implies that  $\mathbf{F}_\ell(\hat{\mathbf{T}}, \underline{E}) = \mathbf{F}_\ell(\tilde{\mathbf{T}}, \underline{E})^T$ , and for some  $\check{A}$ ,  $\check{B}$ ,  $\check{C}$  and  $\check{D}$  we have

$$\mathbf{F}_\ell(\hat{\mathbf{T}}, \underline{E}) : \begin{bmatrix} \hat{x}^+ \\ \hat{r} \end{bmatrix} : \left[ \begin{array}{c|c} \check{A} & \check{B} \\ \check{C} & \check{D} \end{array} \right] : \begin{bmatrix} \hat{x} \\ \hat{n} \end{bmatrix} \quad (\text{B.12})$$

$$\mathbf{F}_\ell(\tilde{\mathbf{T}}, \underline{E}) : \begin{bmatrix} \tilde{x}^+ \\ \tilde{r} \end{bmatrix} : \left[ \begin{array}{c|c} \check{A}^T & \check{C}^T \\ \check{B}^T & \check{D}^T \end{array} \right] : \begin{bmatrix} \tilde{x} \\ \tilde{n} \end{bmatrix} \quad (\text{B.13})$$

Of course  $\mathbf{F}_\ell(\hat{\mathbf{T}}, \underline{E})$  is stable/unstable if and only if  $\mathbf{F}_\ell(\tilde{\mathbf{T}}, \underline{E})^T$  is stable/unstable respectively. Hence if  $\mathbf{H}$  is not MS stable then  $\mathbf{H}^T$  is not MS stable and we have  $\|\mathbf{H}\|_{MSP} = \|\mathbf{H}^T\|_{MSP} = \infty$  by Definition 3. So consider the case where both are MS stable.

Note that if  $\bar{B}_n$  is  $m_n$  wide and  $\bar{C}_r$  is  $l_r$  tall, then  $\hat{n}(k) = \text{vec}(I_{m_n})$  and  $\tilde{n}(k) = \text{vec}(I_{l_r})$ . Moreover,  $\text{tr}(\hat{R}(k)) = \text{vec}(I_{l_r})^T \hat{r}(k)$  and  $\text{tr}(\tilde{R}(k)) = \text{vec}(I_{m_n})^T \tilde{r}(k)$ . Starting from  $\hat{x}(0) = \tilde{x}(0) = 0$  we have that

$$\text{tr}(\hat{R}(k)) = \text{vec}(I_{l_r})^T \hat{r}(k) = \text{vec}(I_{l_r})^T \check{C} \left( \sum_{n=0}^k \check{A}^n \right) \check{B} \text{vec}(I_{m_n}) \quad (\text{B.14})$$

$$\text{tr}(\tilde{R}(k)) = \text{vec}(I_{m_n})^T \tilde{r}(k) = \text{vec}(I_{m_n})^T \check{B}^T \left( \sum_{n=0}^k \check{A}^n \right)^T \check{C}^T \text{vec}(I_{l_r}) \quad (\text{B.15})$$

i.e.  $\text{tr}(\hat{R}(k)) = \text{tr}(\tilde{R}(k))^T = \text{tr}(\tilde{R}(k)) \forall k \geq 0$ . Since  $\hat{R} = \lim_{k \rightarrow \infty} \hat{R}(k)$  and  $\tilde{R} = \lim_{k \rightarrow \infty} \tilde{R}(k)$  are invariant with respect to initial conditions,  $\text{tr}(\hat{R}) = \text{tr}(\tilde{R}) \Rightarrow \|\mathbf{H}\|_{MSP} = \|\mathbf{H}^T\|_{MSP}$ .  $\square$

Before proving Theorem 1 we define a compact notation which will very useful:

**Definition 15.** Given  $\mathbf{H} = \mathbf{F}_\ell(\mathbf{T}, \Delta)$  where  $\mathbf{T}$  has realization (2.2) and  $\Sigma = \text{cov}(\Delta \mathbf{1})$ , let the linear operator  $\mathbf{D} : \mathbf{S}_+^{l_z} \mapsto \mathbf{S}_+^{l_z}$  be given by

$$\mathbf{D}(M) = \bar{D}_{zw} (\Sigma \circ M) \bar{D}_{zw}^T \quad (\text{B.16})$$

**Corollary 14.** Given  $\mathbf{H} = \mathbf{F}_\ell(\mathbf{T}, \Delta)$  where  $\mathbf{T}$  has realization (2.2) and  $\Sigma = \text{cov}(\Delta \mathbf{1})$ , subject to exogenous noise with  $\text{cov}(n(k)) = N$ , let the linear operator  $\mathbf{D}$  be as given in (B.16) and  $X(k)$  and  $R(k)$  be as in Definitions 1 and 2. Then  $\forall k \geq 0$

$$X(k+1) = \bar{A}X(k)\bar{A}^T + \bar{B}_n N \bar{B}_n + \bar{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} \mathbf{D}^m (\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) \right] \right] \bar{B}_w^T \quad (\text{B.17a})$$

$$R(k) = \bar{C}_r X(k) \bar{C}_r^T + \bar{D}_{rn} N \bar{D}_{rn} + \bar{D}_{rw} \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} \mathbf{D}^m (\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) \right] \right] \bar{D}_{rw}^T \quad (\text{B.17b})$$

*Proof of Corollary 14.* From Lemma 1 we have that  $\forall k \geq 0$  there exist  $Z(k) \triangleq \mathbf{E}(z(k)z^T(k))$  and  $W(k) \triangleq \mathbf{E}(w(k)w^T(k))$  such that

$$X(k+1) = \bar{A}X(k)\bar{A}^T + \bar{B}_n N \bar{B}_n^T + \bar{B}_w W(k) \bar{B}_w^T \quad (\text{B.18a})$$

$$R(k) = \bar{C}_r X(k) \bar{C}_r^T + \bar{D}_{rn} N \bar{D}_{rn}^T + \bar{D}_{rw} W(k) \bar{D}_{rw}^T \quad (\text{B.18b})$$

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \bar{D}_{zw} W(k) \bar{D}_{zw}^T \quad (\text{B.18c})$$

$$W(k) = \Sigma \circ Z(k) \quad (\text{B.18d})$$

Assume  $\eta$  is  $q \times 1$ , i.e.  $\Sigma = \text{diag}(\sigma_1^2 \mathbf{1}_{\eta_1 \times \eta_1}, \dots, \sigma_q^2 \mathbf{1}_{\eta_q \times \eta_q}) \in \mathbf{B}_+^\eta$ . Then it can be verified that due to Assumption 5 the operator  $\mathbf{D}$  in (B.16) is nilpotent with degree  $q$ . Therefore, if we recursively substitute  $Z(k)$  in (B.18c) into itself using (B.18d) we have

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \mathbf{D}(Z(k)) \quad (\text{B.19a})$$

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \mathbf{D}(\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) + \mathbf{D}(\mathbf{D}(Z(k))) \quad (\text{B.19b})$$

$\vdots$

$$Z(k) = \bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \sum_{m=1}^{q-1} \mathbf{D}^m (\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) \quad (\text{B.19c})$$

$$Z(k) = \sum_{m=0}^{q-1} \mathbf{D}^m (\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) \quad (\text{B.19d})$$

$$\Rightarrow W(k) = \Sigma \circ \left[ \sum_{m=0}^{q-1} \mathbf{D}^m (\bar{C}_z X(k) \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T) \right] \quad (\text{B.19e})$$

Substituting (B.19e) into (B.18a) and (B.18b) yields (B.17a) and (B.17b).  $\square$

*Proof of Theorem 1.* (a) First we show  $\nu^2 > \|\mathbf{H}\|_{MSP}^2 \Leftrightarrow \exists X \in \mathbf{S}_{++}^n$  and  $R \in \mathbf{S}_{++}^{l_r}$  such that

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m (\bar{C}_z X \bar{C}_z^T + \bar{D}_{zn} \bar{D}_{zn}^T) \right] \right] \bar{B}_w^T \quad (\text{B.20a})$$

$$R \succ \bar{C}_r X \bar{C}_r^T + \bar{D}_{rn} \bar{D}_{rn}^T + \bar{D}_{rw} \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m (\bar{C}_z X \bar{C}_z^T + \bar{D}_{zn} \bar{D}_{zn}^T) \right] \right] \bar{D}_{rw}^T \quad (\text{B.20b})$$

with  $\nu^2 = \text{tr}(R)$ . Then we show the existence of  $X \in \mathbf{S}_{++}^n$  and  $R \in \mathbf{S}_{++}^{l_r}$ , satisfying (B.20a) and (B.20b) is equivalent to the existence of  $X \in \mathbf{S}_{++}^n$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$  and  $W \in \mathbf{B}_{++}^n$  satisfying (3.3a)-(3.3d), i.e. satisfying

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_n N \bar{B}_n^T + \bar{B}_w W \bar{B}_w^T \quad (\text{B.21a})$$

$$R \succ \bar{C}_r X \bar{C}_r^T + \bar{D}_{rn} N \bar{D}_{rn}^T + \bar{D}_{rw} W \bar{D}_{rw}^T \quad (\text{B.21b})$$

$$Z \succ \bar{C}_z X \bar{C}_z^T + \bar{D}_{zn} N \bar{D}_{zn}^T + \bar{D}_{zw} W \bar{D}_{zw}^T \quad (\text{B.21c})$$

$$W \succ \Sigma \circ Z \quad (\text{B.21d})$$

To begin, note that by expanding/rewriting matrix products on the right hand side of Equations (B.17a) and (B.17b) with  $N = I$  there exist some sets of matrices  $\{A_\ell\}, \{B_\ell\}, \{C_\ell\}$ , and  $\{D_\ell\}$  such that

$$X(k+1) = \bar{A}X(k)\bar{A}^T + \bar{B}_n \bar{B}_n^T + \sum_{\ell} A_{\ell} X A_{\ell}^T + B_{\ell} B_{\ell}^T \quad (\text{B.22a})$$

$$R(k) = \bar{C}_r X \bar{C}_r^T + \bar{D}_{rn} \bar{D}_{rn}^T + \sum_{\ell} C_{\ell} X C_{\ell}^T + D_{\ell} D_{\ell}^T \quad (\text{B.22b})$$

From Corollary 14 and Chapter 9 of [60],  $\mathbf{H}$  is MS stable if and only if  $\exists Y \succ 0$  such that

$$Y \succ \bar{A}Y\bar{A}^T + \sum_{\ell} A_{\ell} Y A_{\ell}^T \quad (\text{B.23})$$

Moreover, by definition  $\|\mathbf{H}\|_{MSP}^2 = \bar{\nu}^2$  iff  $\forall X(0) \succeq 0$ , as  $k \rightarrow \infty$   $X(k)$  approaches a unique limit  $\bar{X} \succeq 0$  as  $k \rightarrow \infty$  satisfying

$$\bar{X} = \bar{A}\bar{X}\bar{A}^T + \bar{B}_n \bar{B}_n^T + \sum_{\ell} A_{\ell} \bar{X} A_{\ell}^T + B_{\ell} B_{\ell}^T \quad (\text{B.24a})$$

$$\bar{R} = \bar{C}_r \bar{X} \bar{C}_r^T + \bar{D}_{rn} \bar{D}_{rn}^T + \sum_{\ell} C_{\ell} \bar{X} C_{\ell}^T + D_{\ell} D_{\ell}^T \quad (\text{B.24b})$$

where  $\bar{\nu}^2 = \text{tr}(\bar{R})$

Now assume  $X \succ 0$  and  $R \succ 0$  satisfy (B.20a)-(B.20b). Clearly  $X$  satisfies inequality (B.23) implying MS stability. Therefore there exists a unique  $\bar{X} \succeq 0$  satisfying (B.24a) and (B.24b) with  $\|\mathbf{H}\|_{MSP}^2 = \text{tr}(\bar{R})$ . Subtracting (B.24a) from (B.20a) it is clear that letting  $U \triangleq X - \bar{X}$  there exists a  $V \succ 0$  where

$$U = \bar{A}U\bar{A}^T + \sum_{\ell} A_{\ell}U A_{\ell}^T + V \quad (\text{B.25})$$

The stability of (B.22a) implies the solution to (B.25) can be found by Proposition 14 (page 60) where  $V \succ 0 \Rightarrow U \succ 0 \Rightarrow X \succ \bar{X} \Rightarrow \nu^2 > \|\mathbf{H}\|_{MSP}^2$ .

Conversely, assume  $\|\mathbf{H}\|_{MSP}^2 < \nu^2$ . This implies MS stability by Definition 2, which implies the existence of two things:  $Y \succ 0$  satisfying (B.23), and a unique  $\bar{X} \succeq 0$  satisfying (B.24a) and (B.24b) where  $\|\mathbf{H}\|_{MSP}^2 = \text{tr}(\bar{R})$ . Note that given such a  $Y$

$$\delta Y \succ \bar{A}\delta Y\bar{A}^T + \sum_{\ell} A_{\ell}\delta Y A_{\ell}^T \quad \forall \delta > 0$$

Therefore  $X \triangleq \bar{X} + \delta Y$  satisfies (B.20a) for any  $\delta > 0$ , and for any  $\epsilon > 0 \exists \delta > 0$  s.t.  $\epsilon I \succ \delta Y \Rightarrow \epsilon I \succ X - \bar{X}$ . Therefore there exist  $X$  arbitrarily near  $\bar{X}$ , and hence an  $R$  arbitrarily near  $\bar{R}$  satisfying  $\nu^2 > \text{tr}(R)$ .

Finally we show there exist  $X \in \mathbf{S}_{++}^n$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$  and  $W \in \mathbf{B}_{++}^{\eta}$  satisfying (B.21a)-(B.21d) if and only if there exist  $X \in \mathbf{S}_{++}^n$  and  $R \in \mathbf{S}_{++}^{l_r}$  satisfy (B.20a) and (B.20b). We will first show that inequalities (B.20a) and (B.20b) are feasible with some  $X \succ 0$  and  $R \succ 0$  if and only if there exists a  $Z \succ 0$  such that

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_w(\Sigma \circ Z)\bar{B}_w^T \quad (\text{B.26a})$$

$$R \succ \bar{C}_r X \bar{C}_r^T + \bar{D}_{rn}\bar{D}_{rn}^T + \bar{D}_{rw}(\Sigma \circ Z)\bar{D}_{rw}^T \quad (\text{B.26b})$$

$$Z \succ \bar{C}_z X \bar{C}_z^T + \bar{D}_{zn}\bar{D}_{zn}^T + \bar{D}_{zw}(\Sigma \circ Z)\bar{D}_{zw}^T \quad (\text{B.26c})$$

To see this, assume (B.20a)-(B.20b) are feasible. Recalling the proof of Corollary 14 it can be easily verified that these are exactly

$$X \succ \bar{A}X\bar{A}^T + \bar{B}_n\bar{B}_n^T + \bar{B}_w(\Sigma \circ Z_0)\bar{B}_w^T \quad (\text{B.27a})$$

$$R \succ \bar{C}_rX\bar{C}_r^T + \bar{D}_{rn}\bar{D}_{rn}^T + \bar{D}_{rw}(\Sigma \circ Z_0)\bar{D}_{rw}^T \quad (\text{B.27b})$$

$$Z_0 = \bar{C}_zX\bar{C}_z^T + \bar{D}_{zn}\bar{D}_{zn}^T + \bar{D}_{zw}(\Sigma \circ Z_0)\bar{D}_{zw}^T \quad (\text{B.27c})$$

Let  $Z_\alpha \triangleq \text{diag}(\alpha_1 I_{\eta_1}, \dots, \alpha_q I_{\eta_q}) \succ 0$ . Then for  $i = 1$  to  $q$ , let  $\bar{D}_{zw_i}$  be the  $\eta_i$  columns of  $\bar{D}_{zw}$  multiplying  $\delta_i$ , and define the outer product matrix  $\hat{D}_i \triangleq \bar{D}_{zw_i}\bar{D}_{zw_i}^T$ . This means  $\bar{D}_{zw}(\Sigma \circ Z_\alpha)\bar{D}_{zw}^T = \sum_{i=1}^q \bar{D}_{zw_i}\bar{D}_{zw_i}^T\sigma_i^2\alpha_i = \sum_{i=1}^q \hat{D}_i\sigma_i^2\alpha_i$ . Without loss of generality<sup>1</sup> assume  $\bar{D}_{zw}$  is strictly block lower triangular. This implies that the first  $\sum_{j=1}^i \eta_j$  rows and columns of  $\hat{D}_i$  are zero. In other words, for  $i$  increasing from 1 to  $q$ , a progressively smaller sub-block of the  $\hat{D}_i$  in the lower right is nonzero, until  $\hat{D}_q = 0$  entirely.

Now given any  $\alpha_q > 0$  we can choose  $\alpha_{q-1}$  such that  $Z_\alpha \succ \hat{D}_{q-1}\sigma_{q-1}^2\alpha_{q-1}$  for any  $\{\alpha_i\}_1^{q-2} > 0$ . Similarly we can then choose  $\alpha_{q-2}$  such that  $Z_\alpha \succ \hat{D}_{q-1}\sigma_{q-1}^2\alpha_{q-1} + \hat{D}_{q-2}\sigma_{q-2}^2\alpha_{q-2}$  for any  $\{\alpha_i\}_1^{q-3} > 0$ . We can continue<sup>2</sup> to  $i = 1$  to have  $Z_\alpha \succ \sum_{i=1}^q \hat{D}_i\sigma_i^2\alpha_i = \bar{D}_{zw}(\Sigma \circ Z_\alpha)\bar{D}_{zw}^T$ , which implies  $Z \triangleq Z_0 + Z_\alpha$  satisfies (B.26c). Since we can start from an arbitrarily small  $\alpha_q > 0$ , we can construct such a  $Z$  additionally satisfying (B.26a)-(B.26b) with any given feasible  $X \succ 0$  and  $R \succ 0$ .

The other way is easier. If (B.26a)-(B.26c) are satisfied, then recursively substituting for  $Z$  as we did previously to obtain (B.17a)-(B.17b) gives us (B.20a) and (B.20b).

Finally, the strictness of inequalities (B.26a)-(B.26c) implies they are feasible if and only if there exists some  $W \in \mathbf{B}_{++}^\eta$  which along with the  $X$ ,  $R$ , and  $Z$  in (B.26a)-(B.26c) satisfies (B.21a)-(B.21d), i.e. (3.3a)-(3.3d).

(b) This follows part (a) by duality (Corollary 3).

(c) This follows directly from parts (a) and (b). □

<sup>1</sup>If strictly block upper triangular, the indexing is reversed.

<sup>2</sup>Roughly, we're using the structures of  $Z_\alpha$  and  $\hat{D}_i$  which imply at the  $n^{\text{th}}$  step there is a subsequent  $\alpha_{q-n}$  which can be made arbitrarily small in order to fit  $\hat{D}_{q-n}\sigma_{q-n}^2\alpha_{q-n}$  between  $Z_\alpha$  and  $\sum_{j=q-n}^q \hat{D}_j\sigma_j^2\alpha_j$ .

## APPENDIX C. PROOFS AND AUXILIARY THEORY FOR CHAPTER 5

*Proof of Lemma 2.* (a)  $\mathbf{K}_{\text{FI}} \neq \emptyset \Leftrightarrow \bar{\mathbf{K}}_{\text{FI}} \neq \emptyset$  is obvious since  $\mathbf{K}_{\text{FI}} \supseteq \bar{\mathbf{K}}_{\text{FI}}$ . Motivated by similar scenarios in the literature<sup>1</sup>, to prove  $\mathbf{K}_{\text{FI}} \neq \emptyset \Rightarrow \bar{\mathbf{K}}_{\text{FI}} \neq \emptyset$ , we show given any  $\tilde{\mathbf{K}}_{\text{FI}} \in \mathbf{K}_{\text{FI}}$  we can construct a  $\bar{\mathbf{K}}_{\text{FI}} \in \bar{\mathbf{K}}_{\text{FI}}$  with equivalent or better performance. Any  $\tilde{\mathbf{K}}_{\text{FI}} \in \mathbf{K}_{\text{FI}}$  has a realization

$$\tilde{\mathbf{K}}_{\text{FI}} : \begin{bmatrix} x_{\text{K}}^+ \\ u \end{bmatrix} = \left[ \begin{array}{c|ccc} A_{\text{K}} & B_{\text{K}_x} & B_{\text{K}_n} & B_{\text{K}_w} \\ \hline C_{\text{K}} & D_{\text{K}_x} & D_{\text{K}_n} & 0 \end{array} \right] \begin{bmatrix} x_{\text{K}} \\ x_{\text{G}} \\ n \\ w \end{bmatrix}$$

Let  $\tilde{\mathbf{T}}_{\text{FI}} = \text{F}_\ell(\mathbf{G}_{\text{FI}}, \tilde{\mathbf{K}}_{\text{FI}})$  with realization

$$\tilde{\mathbf{T}}_{\text{FI}} : \begin{bmatrix} x^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} \tilde{A} & \tilde{B}_n & \tilde{B}_w \\ \hline \tilde{C}_r & \tilde{D}_{rn} & D_{rw} \\ \tilde{C}_z & D_{\text{K}_n} & 0 \end{array} \right] \begin{bmatrix} x \\ n \\ w \end{bmatrix} \quad (\text{C.1})$$

where

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} A+B_u D_{\text{K}_x} & B_u C_{\text{K}} \\ B_{\text{K}_x} & A_{\text{K}} \end{bmatrix} & \tilde{B}_n &= \begin{bmatrix} B_n+B_u D_{\text{K}_n} \\ B_{\text{K}_n} \end{bmatrix} & \tilde{B}_w &= \begin{bmatrix} B_w \\ B_{\text{K}_w} \end{bmatrix} \\ \tilde{C}_r &= \begin{bmatrix} C_r+D_r D_{\text{K}_x} & D_r C_{\text{K}} \end{bmatrix} & \tilde{D}_{rn} &= D_{rn}+D_r D_{\text{K}_n} \\ \tilde{C}_z &= \begin{bmatrix} D_{\text{K}_x} & C_{\text{K}} \end{bmatrix} \end{aligned}$$

and  $\tilde{\mathbf{H}}_{\text{FI}} = \text{F}_\ell(\tilde{\mathbf{T}}_{\text{FI}}, \Delta)$ .

<sup>1</sup>See e.g. Proof of Theorem 4.1 in [69] or Notes for Chapter 7 in [60].

By Theorem 1 there exist  $X \in \mathbf{S}_{++}^{n\tau}$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  such that

$$X \succ \tilde{A}X\tilde{A}^T + \tilde{B}_n\tilde{B}_n^T + \tilde{B}_wW\tilde{B}_w^T \quad (\text{C.2a})$$

$$R \succ \tilde{C}_rX\tilde{C}_r^T + \tilde{D}_{rn}\tilde{D}_{rn}^T + D_{rw}WD_{rw}^T \quad (\text{C.2b})$$

$$Z \succ \tilde{C}_zX\tilde{C}_z^T + D_{\kappa_n}D_{\kappa_n}^T \quad (\text{C.2c})$$

$$W \succ \Sigma \circ Z \quad (\text{C.2d})$$

with  $\|\tilde{\mathbf{H}}_{\text{Fi}}\|_{\text{MSP}}^2 < \text{tr}(R)$ . Taking any feasible solution, partition  $X$  by the dimensions of the plant and controller states as:

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \quad (\text{C.3})$$

Expanding the  $X_1$  block of inequality (C.2a) we have:

$$\begin{aligned} X_1 \succ & (A + B_uD_{\kappa_x})X_1(A + B_uD_{\kappa_x})^T + (B_uC_{\kappa})X_3(B_uC_{\kappa})^T \\ & + (A + B_uD_{\kappa_x})X_2(B_uC_{\kappa})^T + (B_uC_{\kappa})X_2^T(A + B_uD_{\kappa_x})^T \\ & + (B_n + B_uD_{\kappa_n})(B_n + B_uD_{\kappa_n})^T + B_wWB_w^T \end{aligned} \quad (\text{C.4})$$

Expanding (C.2b) gives:

$$\begin{aligned} R \succ & (C_r + D_rD_{\kappa_x})X_1(C_r + D_rD_{\kappa_x})^T + (D_rC_{\kappa})X_3(D_rC_{\kappa})^T \\ & + (C_r + D_rD_{\kappa_x})X_2(D_rC_{\kappa})^T + (D_rC_{\kappa})X_2^T(C_r + D_rD_{\kappa_x})^T \\ & + (D_{rn} + D_rD_{\kappa_n})(D_{rn} + D_rD_{\kappa_n})^T + D_{rw}WD_{rw}^T \end{aligned} \quad (\text{C.5})$$

And similarly expanding (C.2c):

$$Z \succ D_{\kappa_x}X_1D_{\kappa_x}^T + D_{\kappa_x}X_2C_{\kappa}^T + C_{\kappa}X_2^TD_{\kappa_x}^T + C_{\kappa}X_3C_{\kappa}^T + D_{\kappa_n}D_{\kappa_n}^T \quad (\text{C.6})$$

Now we restrict our attention to static feedback. Any static controller  $\bar{\mathbf{K}}_{\text{Fi}} \in \mathbf{K}_{\text{Fi}}$  must have the structure

$$\bar{\mathbf{K}}_{\text{Fi}} : u = \begin{bmatrix} F & F_0 & 0 \end{bmatrix} \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.7})$$

and result in a mean closed loop  $\bar{\mathbf{T}}_{\text{FI}}$ :

$$\bar{\mathbf{T}}_{\text{FI}}: \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A_F & B_{F_0} & B_w \\ \hline C_F & D_{F_0} & D_{rw} \\ F & F_0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.8})$$

where  $A_F = A + B_u F$ ,  $B_{F_0} = B_n + B_u F_0$ ,  $C_F = C_r + D_r F$ ,  $D_{F_0} = D_{rn} + D_r F_0$ , and  $\bar{\mathbf{H}}_{\text{FI}} = F_\ell(\bar{\mathbf{T}}_{\text{FI}}, \Delta)$ . If we form  $F = D_{\kappa_x} + C_\kappa X_2^T X_1^{-1}$  using any feasible  $X$  from (C.2a)-(C.2d) partitioned as in (C.3), and  $F_0 = D_{\kappa_n}$ , then for some  $\bar{Z} \preceq Z$  and  $\bar{W} \preceq W$ , and  $\bar{R} \preceq R$  we have:

$$X_1 \succ A_F X_1 A_F^T + B_{F_0} B_{F_0}^T + B_w \bar{W} B_w^T \quad (\text{C.9a})$$

$$\bar{R} \succ C_F X_1 C_F^T + D_{F_0} D_{F_0}^T + D_{rw} \bar{W} D_{rw}^T \quad (\text{C.9b})$$

$$\bar{Z} \succ F X_1 F^T + F_0 F_0^T \quad (\text{C.9c})$$

$$\bar{W} \succ \Sigma \circ \bar{Z} \quad (\text{C.9d})$$

This implies  $\bar{\mathbf{K}}_{\text{FI}} \in \mathbf{K}_{\text{FI}}$  and  $\|\bar{\mathbf{H}}_{\text{FI}}\|_{\text{MSP}}^2 < \text{tr}(\bar{R})$  by Theorem 1. To verify that this is true, note that by substituting  $F = D_{\kappa_x} + C_\kappa X_2^T X_1^{-1}$  and  $F_0 = D_{\kappa_n}$  into (C.9a)-(C.9c) gives:

$$\begin{aligned} X_1 \succ & (A + B_u D_{\kappa_x}) X_1 (A + B_u D_{\kappa_x})^T + (B_u C_\kappa) X_2^T X_1^{-1} X_2 (B_u C_\kappa)^T \\ & + (A + B_u D_{\kappa_x}) X_2 (B_u C_\kappa)^T + (B_u C_\kappa) X_2^T (A + B_u D_{\kappa_x})^T \\ & + (B_n + B_u D_{\kappa_n}) (B_n + B_u D_{\kappa_n})^T + B_w \bar{W} B_w^T \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned} \bar{R} \succ & (C_r + D_r D_{\kappa_x}) X_1 (C_r + D_r D_{\kappa_x})^T + (D_r C_\kappa) X_2^T X_1^{-1} X_2 (D_r C_\kappa)^T \\ & + (C_r + D_r D_{\kappa_x}) X_2 (D_r C_\kappa)^T + (D_r C_\kappa) X_2^T (C_r + D_r D_{\kappa_x})^T \\ & + (D_{rn} + D_r D_{\kappa_n}) (D_{rn} + D_r D_{\kappa_n})^T + D_{rw} \bar{W} D_{rw}^T \end{aligned} \quad (\text{C.11})$$

and

$$\bar{Z} \succ D_{\kappa_x} X_1 D_{\kappa_x}^T + D_{\kappa_x} X_2 C_\kappa^T + C_\kappa X_2^T D_{\kappa_x}^T + C_\kappa X_2^T X_1^{-1} X_2 C_\kappa^T + D_{\kappa_n} N D_{\kappa_n}^T \quad (\text{C.12})$$

Now examining (C.10), (C.11), and (C.12), we can easily verify that they are the same as those in (C.4), (C.5), and (C.6) except for having  $X_2^T X_1^{-1} X_2$  in place of  $X_3$ . Because  $X \succ 0 \Rightarrow X_3 \succ X_2^T X_1^{-1} X_2$ , we know we can allow  $\bar{Z} \preceq Z$  and  $\bar{W} \preceq W$ , and  $\bar{R} \preceq R$ . Note that we can choose  $\bar{R} \neq R \Rightarrow \text{tr}(\bar{R}) < \text{tr}(R)$ . This will be relevant for (c).



To show  $\bar{\mathbf{K}}_{\mathbf{F}\mathbf{I}} \neq \emptyset \Leftrightarrow \mathbf{M}_{\mathbf{F}} \neq \emptyset$ , note any static controller  $\bar{\mathbf{K}}_{\mathbf{F}\mathbf{I}}$  will produce a mean closed loop with the structure (C.8). By Theorem 1  $\bar{\mathbf{K}}_{\mathbf{F}\mathbf{I}} \in \bar{\mathbf{K}}_{\mathbf{F}\mathbf{I}}$  if and only if there exists a  $X \in \mathbf{S}_{++}^n$ ,  $R \in \mathbf{S}_{++}^l$ ,  $Z \in \mathbf{S}_{++}^l$ ,  $W \in \mathbf{B}_{++}^\eta$  such that

$$X \succ (A + B_u F)X(A + B_u F)^T + (B_n + B_u F_0)(B_n + B_u F_0)^T + \tilde{B}_w W \tilde{B}_w^T \quad (\text{C.13a})$$

$$R \succ (C_r + D_r F)X(C_r + D_r F)^T + (D_{rn} + D_r F_0)(D_{rn} + D_r F_0)^T + D_{rw} W D_{rw}^T \quad (\text{C.13b})$$

$$Z \succ F X F^T + F_0 F_0^T \quad (\text{C.13c})$$

$$W \succ \Sigma \circ Z \quad (\text{C.13d})$$

Notice that (C.13a)-(C.13d) are trivially equivalent to

$$\begin{aligned} X - (A + B_u F)X X^{-1}X(A + B_u F)^T - (B_n + B_u F_0)(B_n + B_u F_0)^T - \tilde{B}_w W W^{-1}W \tilde{B}_w^T &\succ 0 \\ R - (C_r + D_r F)X X^{-1}X(C_r + D_r F)^T - (D_{rn} + D_r F_0)(D_{rn} + D_r F_0)^T - D_{rw} W W^{-1}W D_{rw}^T &\succ 0 \\ Z - F X X^{-1}X F^T - F_0 F_0^T &\succ 0 \\ W - \Sigma \circ Z &\succ 0 \end{aligned}$$

Applying Schur complement and the change of variables  $Y = FX$  and  $J = F_0$  we to these inequalities we obtain LMIs:

$$\begin{bmatrix} X & (AX+B_u Y) & (B_n+B_u J) & B_w W \\ (AX+B_u Y)^T & X & 0 & 0 \\ (B_n+B_u J)^T & 0 & I & 0 \\ (B_w W)^T & 0 & 0 & W \end{bmatrix} \succ 0 \quad (\text{C.15a})$$

$$\begin{bmatrix} R & (CX+D_r Y) & (D_1+D_r J) & D_2 W \\ (CX+D_r Y)^T & X & 0 & 0 \\ (D_{rn}+D_r J)^T & 0 & I & 0 \\ (D_{rw} W)^T & 0 & 0 & W \end{bmatrix} \succ 0 \quad (\text{C.15b})$$

$$\begin{bmatrix} Z & Y & J \\ Y^T & X & 0 \\ J^T & 0 & I \end{bmatrix} \succ 0 \quad (\text{C.15c})$$

$$W - \Sigma \circ Z \succ 0 \quad (\text{C.15d})$$

(b) This follows from the proof of part (a) above.

(c) From the definition of  $\mathbf{K}_{\mathbf{F}_I}$  and the proof of part (a) we know that there exist  $\tilde{\mathbf{K}}_{\mathbf{F}_I} \in \mathbf{K}_{\mathbf{F}_I}$  and  $\bar{\mathbf{K}}_{\mathbf{F}_I} \in \bar{\mathbf{K}}_{\mathbf{F}_I}$  such that  $(\nu_{\mathbf{F}_I}^*)^2 < \text{tr}(\bar{\mathbf{R}}) < \text{tr}(\mathbf{R}) < (\nu_{\mathbf{F}_I}^*)^2 + \epsilon$  for any  $\epsilon > 0$ . Therefore this follows from part (b). □

*Proof of Corollary 6.* By Lemma 2 we know that  $\mathbf{K}_{\mathbf{F}_I} \neq \emptyset \Leftrightarrow \bar{\mathbf{K}}_{\mathbf{F}_I} \neq \emptyset$ . Now by Corollary 5  $\bar{\mathbf{K}}_{\mathbf{F}_I} \neq \emptyset$  if and only if there exist a  $F \in \mathbf{R}^{m_u \times n}$ ,  $X \in \mathbf{S}_{++}^n$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  such that

$$X \succ (A + B_u F)X(A + B_u F)^T + \tilde{B}_w W \tilde{B}_w^T \quad (\text{C.16a})$$

$$Z \succ F X F^T \quad (\text{C.16b})$$

$$W \succ \Sigma \circ Z \quad (\text{C.16c})$$

This is equivalent to the existence of  $X \in \mathbf{S}_{++}^n$ ,  $Y \in \mathbf{R}^{m_u \times n}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  satisfying

$$\begin{bmatrix} X & (AX+B_u Y) & B_w W \\ (AX+B_u Y)^T & X & 0 \\ (B_w W)^T & 0 & W \end{bmatrix} \succ 0 \quad (\text{C.17a})$$

$$\begin{bmatrix} Z & Y \\ Y^T & X \end{bmatrix} \succ 0 \quad (\text{C.17b})$$

$$W - \Sigma \circ Z \succ 0 \quad (\text{C.17c})$$

with the change of variables  $Y = FX$ . Therefore  $\mathbf{K}_{\mathbf{F}_I} \neq \emptyset$  if and only if there exist  $X \in \mathbf{S}_{++}^n$ ,  $Y \in \mathbf{R}^{m_u \times n}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$ ,  $W \in \mathbf{B}_{++}^\eta$  satisfying (C.17a)-(C.17c), i.e. (5.5a)-(5.5c).

Finally, recalling Definition 5 it is straightforward to verify that the feasibility of (C.16a)-(C.16a) is equivalent to that of

$$X \succ (A+B_u F)X(A+B_u F)^T + B_w(\Sigma \circ (F X F^T))B_w^T = \mathbf{A}_F(X) \quad (\text{C.18})$$

which by Propositions 12 and 15 is equivalent to  $\rho(\mathbf{A}_F) < 1$ . □

Proposition 16 will be used in the proof of Lemma 3 which follows.

**Proposition 16.** If  $f : \mathbf{R}^{m \times n} \mapsto \mathbf{S}_+^n$  is given by  $f(Y) = \sum_{i=1}^q (A_i + B_i Y)^T M_i (A_i + B_i Y)$  with  $M_i \succeq 0$  and  $\sum_{i=1}^q B_i^T M_i B_i \succ 0$ , then there exists a unique  $Y^*$  such that  $f(Y^*) \preceq f(Y)$  and  $\text{tr}(f(Y^*)) < \text{tr}(f(Y)) \forall Y \neq Y^*$  given by  $Y^* = -(\sum_{i=1}^q B_i^T M_i B_i)^{-1} (\sum_{i=1}^q B_i^T M_i A_i)$ .

*Proof.* Substituting  $Y^*$ , one can easily verify that the cross terms cancel and  $f(Y^* + P) = f(Y^*) + \sum_{i=1}^q P^T B_i^T M_i B_i P$  for any  $P \in \mathbf{R}^{m \times n}$ . Additionally,  $\sum_{i=1}^q B_i M_i B_i^T \succ 0$  implies  $\sum_{i=1}^q P^T B_i^T M_i B_i P \succeq 0$ . Therefore  $f(Y^*) \preceq f(Y) \forall Y \neq Y^*$ . Finally,  $C \succeq 0$  implies  $\text{tr}(C) \geq 0$  and  $\text{tr}(C) = 0 \Leftrightarrow C = 0$ . Therefore  $\text{tr}(f(Y^*)) < \text{tr}(f(Y^* + P))$  for any  $P \neq 0$ , or equivalently,  $\text{tr}(f(Y^*)) < \text{tr}(f(Y)) \forall Y \neq Y^*$ .  $\square$

*Proof of Lemma 3.* By Lemma 2, to investigate the limits of MS performance for Problem 2, we need only consider static controllers (5.4) which produce closed loops  $\mathbf{H}_{\text{FI}} = \mathbf{F}_\ell(\mathbf{T}_{\text{FI}}, \Delta)$  where  $\mathbf{T}_{\text{FI}}$  has the realization (C.19).

$$\mathbf{T}_{\text{FI}}: \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A_F & B_{F_0} & B_w \\ \hline C_F & D_{F_0} & D_{rw} \\ F & F_0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.19})$$

with  $A_F = A + B_u F$ ,  $B_{F_0} = B_n + B_u F_0$ ,  $C_F = C_r + D_r F$ ,  $D_{F_0} = D_{rn} + D_r F_0$ . By Theorem 1,  $\|\mathbf{H}_{\text{FI}}\|_{\text{MSP}} < \text{tr}(R)$  for some  $R \in \mathbf{S}_{++}^{mn}$  if and only if there are  $X \in \mathbf{S}_{++}^n$ ,  $Z \in \mathbf{S}_{++}^{lz}$  and  $W \in \mathbf{B}_{++}^{\eta}$  such that

$$X \succ A_F^T X A_F + C_F^T C_F + F^T W F \quad (\text{C.20a})$$

$$R \succ B_{F_0}^T X B_{F_0} + D_{F_0}^T D_{F_0} + F_0^T W F_0 \quad (\text{C.20b})$$

$$Z \succ B_w^T X B_w + \bar{D}_{rw}^T \bar{D}_{rw} \quad (\text{C.20c})$$

$$W \succ \Sigma \circ Z \quad (\text{C.20d})$$

Since  $D_{zw} = 0$  in (C.19), these LMIs are equivalent to (C.21a)-(C.21b).

$$X \succ A_F^T X A_F + C_F^T C_F + F^T \Sigma \circ (B_w^T X B_w + D_{rw}^T D_{rw}) F \quad (\text{C.21a})$$

$$R \succ B_{F_0}^T X B_{F_0} + D_{F_0}^T D_{F_0} + F_0^T \Sigma \circ (B_w^T X B_w + D_{rw}^T D_{rw}) F_0 \quad (\text{C.21b})$$

Note that since  $D_r^T D_r + \Sigma \circ D_{rw}^T D_{rw} \succ 0 \Rightarrow B_u^T X B_u + D_r^T D_r + \Sigma \circ (B_w^T X B_w + D_{rw}^T D_{rw}) \succ 0$ , (C.21a) and (C.21b) satisfy the structure and assumptions of Proposition 16 with  $Y = F$  and  $Y = F_0$  respectively. This implies (C.21a) and (C.21b) are feasible for some  $F$  and  $F_0$  iff they are feasible for

$$F = -S^{-1}(B_u^T X A) \quad (\text{C.22a})$$

$$F_0 = -S^{-1}(B_u^T X B_n + D_r^T D_{rn}) \quad (\text{C.22b})$$

with  $S = B_u^T X B_u + D_r^T D_r + \Sigma \circ (B_w^T X B_w + D_{rw}^T D_{rw})$ . Substituting (C.22a)-(C.22a) into (C.21a)-(C.21b) produces

$$X \succ A^T X A + C_r^T C_r - (A^T X B_u) S^{-1} (B_u^T X A) \quad (\text{C.23a})$$

$$R \succ B_n^T X B_n + D_{rn}^T D_{rn} - (B_n^T X B_u + D_{rn}^T D_r) S^{-1} (B_u^T X B_n + D_r^T D_{rn}) \quad (\text{C.23b})$$

That is, (C.21a)-(C.21b) are feasible in  $X$ ,  $F$ ,  $F_0$ , and  $R$ , iff (C.23a)-(C.23b) are feasible in  $X$  and  $R$ . Therefore  $\nu_{FI}^* = \inf_{(\text{C.23a})-(\text{C.23b})} \text{tr}(R)^{\frac{1}{2}}$ .

For the remainder of this proof let us define a compact notation: let  $F : \mathbf{S}_+^n \mapsto \mathbf{R}^{m_u \times n}$ ,  $F_0 : \mathbf{S}_+^n \mapsto \mathbf{R}^{m_u \times m_n}$ ,  $R : \mathbf{S}_+^n \mapsto \mathbf{S}_+^n$ , and  $P : \mathbf{S}_+^n \mapsto \mathbf{S}_+^{m_n}$  be given by

$$F(X) = -S^{-1}(B_u^T X A) \quad (\text{C.24a})$$

$$F_0(X) = -S^{-1}(B_u^T X B_n + D_r^T D_{rn}) \quad (\text{C.24b})$$

$$R(X) = A^T X A + C_r^T C_r - (A^T X B_u) S^{-1} (B_u^T X A) \quad (\text{C.24c})$$

$$P(X) = B_n^T X B_n + D_{rn}^T D_{rn} - (B_n^T X B_u + D_{rn}^T D_r) S^{-1} (B_u^T X B_n + D_r^T D_{rn}) \quad (\text{C.24d})$$

respectively, where  $S = B_u^T X B_u + D_r^T D_r + \Sigma \circ (B_w^T X B_w + D_{rw}^T D_{rw})$ .

Under the given conditions, Theorem 6.9 of [67] implies there exists a unique  $X^* \succeq 0$  such that  $X^* = R(X^*)$  and  $[F(X^*) F_0(X^*) 0] \in \mathbf{K}_{FI}$ . Let  $R^* = P(X^*)$ . We will next complete the proof by showing that  $X \succ X^*$  and  $R \succ R^*$  for any  $X$  and  $R$  feasible in (C.23a)-(C.23b), meaning  $\nu_{FI}^* = \text{tr}(R^*)^{\frac{1}{2}}$ .

For  $i = 1, 2$ , let  $X_i \succeq 0$ ,  $F_i = F(X_i)$ , and  $F_{0_i} = F_0(X_i)$ . Now if we further let  $H = X_2 - X_1$ ,  $G_x = F_2 - F_1$ ,  $G_r = F_{0_2} - F_{0_1}$ ,  $P_x = R(X_2) - R(X_1)$ , and  $P_r = P(X_2) - P(X_1)$ , then under the assumptions of Lemma 3, Lemma 5.3 in [67] implies that we have

$$\begin{aligned} P_x &= (A + B_u F_2)^T H (A + B_u F_2) + F_2^T (\Sigma \circ B_w^T H B_w) F_2 + Q_x \\ Q_x &= G_x^T (B_u^T X_1 B_u + D_r^T D_r + \Sigma \circ (B_w^T X_1 B_w + D_{rw}^T D_{rw})) G_x \\ P_r &= (B_n + B_u F_{0_2})^T H (B_n + B_u F_{0_2}) + F_{0_2}^T (\Sigma \circ B_w^T H B_w) F_{0_2} + Q_r \\ Q_r &= G_r^T (B_u^T X_1 B_u + D_r^T D_r + \Sigma \circ (B_w^T X_1 B_w + D_{rw}^T D_{rw})) G_r \end{aligned}$$

We derive two things from this. First,  $X_2 \succeq X_1 \Leftrightarrow H \succeq 0$ , and since  $Q_x \succeq 0$  and  $Q_r \succeq 0$  this implies  $P_x \succeq 0 \Leftrightarrow R(X_2) \succeq R(X_1)$  and  $P_r \succeq 0 \Leftrightarrow P(X_2) \succeq P(X_1)$ .

Second, if  $X_2 = X \succ R(X)$  and  $X_1 = X^* = R(X^*)$ , then  $H \succ P_x$ . Since  $Q_x \succeq 0$  this implies there exists a  $\tilde{Q} \succ 0$  such that  $H = P_x + \tilde{Q}$ . Equivalently,

$$H = (A + B_u F_2)^T H (A + B_u F_2) + F_2^T (\Sigma \circ B_w^T H B_w) F_2 + \tilde{Q} \quad (\text{C.25})$$

for some  $\tilde{Q} = Q_x + \tilde{Q} \succ 0$ . Since  $F_2$  is MS stabilizing, (C.25) has a unique positive definite solution (Thm. 3.3(iv) [67]). I.e.  $H \succ 0 \Leftrightarrow X \succ X^*$ . Finally, if  $R \succ P(X)$  and  $R^* = P(X^*)$ , then  $R - R^* \succ P_r$  and  $H \succ 0$  implies  $P_r \succeq 0$ . Therefore  $R \succ R^*$ .

(b) This follows from Theorem 7.3 in [67].  $\square$

*Proof of Proposition 2.* Consider the modification of the networked FI synthesis in Problem 3 where we seek to minimize  $\|\tilde{H}_{\text{FI}}\|_{\text{MSP}}$  where  $\tilde{H}_{\text{FI}} = H_{\text{FI}} N^{\frac{1}{2}}$  for some  $N \succ 0$ . That is, we want to find the  $F$  and  $F_0$  which minimize  $\|F_{\ell}(\tilde{T}_{\text{FI}}, \Delta)\|_{\text{MSP}}$  where

$$\tilde{T}_{\text{FI}}: \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A+B_u F & B_n N^{\frac{1}{2}} + B_u F_0 N^{\frac{1}{2}} & B_w \\ \hline C_r + D_r F & D_{rn} N^{\frac{1}{2}} + D_r F_0 N^{\frac{1}{2}} & D_{rw} \\ F^* & F_0 N^{\frac{1}{2}} & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.26})$$

Since  $N \succ 0$  we have that  $N^{\frac{1}{2}} \succ 0$ , and therefore  $N^{-\frac{1}{2}}$  exists. Therefore we can equivalently search for the  $F$  and  $\tilde{F}_0$  which minimize  $\|F_{\ell}(\hat{T}_{\text{FI}}, \Delta)\|_{\text{MSP}}$  where

$$\hat{T}_{\text{FI}}: \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{c|cc} A+B_u F & \tilde{B}_n + B_u \tilde{F}_0 & B_w \\ \hline C_r + D_r F & \tilde{D}_{rn} + D_r \tilde{F}_0 & D_{rw} \\ F & \tilde{F}_0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.27})$$

with  $\tilde{B}_n = B_n N^{\frac{1}{2}}$  and  $\tilde{D}_{rn} = D_{rn} N^{\frac{1}{2}}$ . Applying Lemma 3 to this problem, one can easily verify that the optimal gains are  $F^*$  and  $\tilde{F}_0^* = F_0^* N^{\frac{1}{2}}$ , meaning  $F^*$  and  $F_0^*$  minimize  $\|\tilde{H}_{F_1}\|_{MSP}$ .  $\square$

*Proof of Lemma 4.* (a) By inspection of Problems 2 and 3 it is clear that  $\mathbf{K}_{DF} \neq \emptyset$  only if  $\mathbf{K}_{FI} \neq \emptyset \Leftrightarrow \mathbf{M}_F \neq \emptyset$ . To see that  $\mathbf{M}_F \neq \emptyset$  implies  $\mathbf{K}_{DF} \neq \emptyset$ , recall  $\mathbf{K}_{DF}$  in (5.14) and let  $\mathbf{T}_{DF} = F_\ell(\mathbf{G}_{DF}, \mathbf{K}_{DF})$ . It is easy to verify using the coordinates  $x_G(k)$  and  $e(k) = x_G(k) - x_K(k)$  that  $\mathbf{T}_{DF}$  has the realization

$$\mathbf{T}_{DF}: \begin{bmatrix} x_G^+ \\ e^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{cc|cc} A+B_u F & B_u F_0 C_y - B_u F & B_n + B_u F_0 & B_w \\ 0 & A - B_n C_y & 0 & 0 \\ \hline C_r + D_r F & D_r F_0 C_y - D_r F & D_{rn} + D_r F_0 & D_{rw} \\ F & F_0 C_y - F & F_0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ e \\ n \\ w \end{bmatrix} \quad (\text{C.28})$$

Recalling that  $A - B_n C_y$  is Schur, the states  $e(k)$  are stable as well as uncontrollable (i.e. they are non-minimal) and therefore can be removed. Removing  $e(k)$  leaves us with

$$\mathbf{T}_{DF}: \begin{bmatrix} x_G^+ \\ r \\ z \end{bmatrix} = \left[ \begin{array}{cc|cc} A+B_u F & B_n + B_u F_0 & B_w \\ \hline C_r + D_r F & D_{rn} + D_r F_0 & D_{rw} \\ F & F_0 & 0 \end{array} \right] \begin{bmatrix} x_G \\ n \\ w \end{bmatrix} \quad (\text{C.29})$$

In other words  $\mathbf{T}_{DF} = \mathbf{T}_{FI}$  in (C.8). Therefore  $\mathbf{K}_{DF} \neq \emptyset$  if  $\mathbf{K}_{FI} \neq \emptyset \Leftrightarrow \mathbf{M}_F \neq \emptyset$ .

(b) and (c) follow from the proof of (a) and Lemma 2.  $\square$

*Proof of Lemma 5.* The standard  $\mathcal{H}_2$  Riccati solution  $\hat{X}$  exists by the assumptions Lemma 3 as a special case. Expanding  $\hat{X}$  we have

$$\hat{X} = A^T \hat{X} A + C_r^T C_r - (A^T \hat{X} B_u) S^{-1} (B_u^T \hat{X} A) \quad (\text{C.30})$$

where  $S = B_u^T \hat{X} B_u + D_r^T D_r + V$ . We can expand  $X^*$  as

$$X^* = A^T X^* A + C_r^T C_r - (A^T X^* B_u) S^{-1} (B_u^T X^* A) \quad (\text{C.31})$$

where  $S = B_u^T X^* B_u + D_r^T D_r + V$ . Therefore  $X^*$  is a solution to (C.30). Moreover, under the assumed conditions, for a given  $V \succeq 0$  the positive semidefinite solution to (C.30) is unique. Therefore  $X^* = \hat{X}$ . This implies  $F^* = \hat{F}$  and  $F_0^* = \hat{F}_0$ . Finally it is known that  $\gamma_{FI}^* = \gamma_{DF}^* = \text{tr}(\hat{R})$  where  $\hat{R} = B_n^T \hat{X} B_n + D_{rn}^T D_{rn} - (B_n^T \hat{X} B_u + D_{rn}^T D_r) S^{-1} (B_u^T \hat{X} B_n + D_r^T D_{rn})$ .

It can be easily verified that  $X^* = \hat{X} \Rightarrow \hat{R} = R^*$  meaning  $\nu_{FI}^* = \gamma_{FI}^* = \nu_{DF}^* = \gamma_{DF}^*$ .  $\square$

## APPENDIX D. PROOFS AND AUXILIARY THEORY FOR CHAPTER 6

*Proof of Proposition 9.* (a) This follows by applying Theorem 5.4.1 in [66] to  $G_{OF}$  in (6.1) with  $B_2 = [0 \ B_u]$ ,  $C_2 = \begin{bmatrix} C_y \\ 0 \end{bmatrix}$ , and stabilizing gains  $\tilde{F} = \begin{bmatrix} -C_z \\ F \end{bmatrix}$  and  $\tilde{L} = [L \ B_w]$ .

(b) This is a trivial consequence of (a).

(c) This follows from (b) and Corollary 4. □

**Remark 6.** Note  $A+B_2\tilde{F} = A+[0 \ B_u] \begin{bmatrix} -C_z \\ F \end{bmatrix} = A+B_u F$  is Schur with appropriate  $F$ . The  $-C_z$  element does not affect the product  $B_2\tilde{F}$  and therefore does not restrict the parameterization. Similarly  $A+\tilde{L}C_2 = A+[L \ -B_w] \begin{bmatrix} C_y \\ 0 \end{bmatrix} = A+LC_y$  is Schur independently of  $-B_w$ . In fact we could have chosen any matrices of appropriate size in place of  $-C_z$  and  $-B_w$ . However, we will see that using these matrices leads to a structure in the resulting closed loops which is advantageous.

*Proof of Proposition 10.* (a) Using the coordinates  $x_G$  and  $e(k) = x_G(k) - \hat{x}(k)$  to represent the interconnection of (6.7) and (6.8) in Figure 6.2, one can verify the states  $e(k)$  are stable and unobservable and hence can be removed to yield (6.1).

(b) This is a consequence of (a) and Proposition 9. □

*Proof of Proposition 11.* Recall the realization of  $T_{OF}$  in Proposition 10, Figure 6.2, and note that  $z_s(k) = z_{s_e}(k) + z_{s_{\hat{x}}}(k)$ . If we let the star-product<sup>1</sup> of  $G_{DF}$  and  $J$  be  $\hat{U}_{DF} = \text{Star}(G_{DF}, J)$ , then it has realization

$$\hat{U}_{DF} : \begin{bmatrix} \hat{x}^+ \\ r_{\hat{x}} \\ z_{s_{\hat{x}}} \\ z_a \\ s \\ w_a \end{bmatrix} = \left[ \begin{array}{c|cccc} A+B_u F & -L & B_w & 0 & B_u \\ \hline C_r+D_r F & -L_0 & D_{rw} & 0 & D_r \\ 0 & 0 & 0 & I & 0 \\ F & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \end{array} \right] \begin{bmatrix} \hat{x} \\ s \\ w_a \\ q_1 \\ q_2 \end{bmatrix} \quad (D.1)$$

With the previously mentioned choice of  $\tilde{F} = \begin{bmatrix} -C_z \\ F \end{bmatrix}$  in (6.3) (see Remark 6), the signal  $z_{s_{\hat{x}}}(k) = q_1(k)$  in (D.1), meaning  $z_s(k) = z_{s_e}(k) + q_1(k)$ . Moreover  $q_1(k)$  is simply a feed through in (D.1). Therefore we can move it out of  $\hat{U}_{DF}$  giving  $U_{DF}$  in (6.9). Then if we partition  $Q$  as in the statement we have  $q_1(k) = q_{1_s}(k) + q_{1_w}(k)$ .  $\square$

*Proof of Lemma 10.* Given  $H_{OF} = F_{\ell}(\mathcal{T}_{OF}(F, L, L_0, Q), \Delta) \in \mathbf{H}_{OF}$ , partition  $Q$  as in Proposition 11. Since  $Q$  is stable, we can realize  $Q_{11}$ ,  $Q_{12}$  and  $Q_2$  separately. This can always be done in a nonminimal but stable way. Let each subsystem have state  $x_{Q_{11}}$ ,  $x_{Q_{12}}$ , and  $x_{Q_2}$  respectively, and let  $Q_{12}$  have realization

$$Q_{12} : \begin{bmatrix} x_{Q_{12}}^+ \\ q_{1_w} \end{bmatrix} \left[ \begin{array}{c|c} A_{Q_{12}} & B_{Q_{12}} \\ \hline C_{Q_{12}} & D_{Q_{12}} \end{array} \right] \begin{bmatrix} x_{Q_{12}} \\ w_a \end{bmatrix} \quad (D.2)$$

Let a realization of  $T_{OF} = \mathcal{T}_{OF}(F, L, L_0, Q)$  be

$$T_{OF} : \begin{bmatrix} x^+ \\ r \\ z \end{bmatrix} \left[ \begin{array}{c|cc} \tilde{A} & \tilde{B}_n & \tilde{B}_w \\ \hline \tilde{C}_r & \tilde{D}_{rn} & \tilde{D}_{rw} \\ \tilde{C}_z & \tilde{D}_{zn} & \tilde{D}_{zw} \end{array} \right] \begin{bmatrix} x \\ n \\ w \end{bmatrix} \quad (D.3)$$

following Figure 6.3 and recall  $\Sigma = \text{cov}(\Delta 1)$ . Referring to (6.7), (6.9), and Figure 6.3, it can be verified that if we define

$$x_1 = \begin{bmatrix} e \\ x_{Q_{11}} \end{bmatrix} \quad (D.4a)$$

$$x_2 = \begin{bmatrix} \hat{x} \\ x_{Q_2} \end{bmatrix} \quad (D.4b)$$

<sup>1</sup>See e.g. [62] or Chapter 10 of [61] for a formal description of the star-product feedback interconnection.



and let  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_{Q12} \end{bmatrix}$  then (D.3) has the expanded structure

$$\mathbf{T}_{\text{OF}} : \begin{bmatrix} x_1^+ \\ x_2^+ \\ x_{Q12}^+ \\ r \\ z_s \\ z_a \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & 0 & 0 & \tilde{B}_{n_1} & \tilde{B}_{w11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & 0 & \tilde{B}_{n_2} & \tilde{B}_{w12} & \tilde{B}_{w22} \\ 0 & 0 & \tilde{A}_{Q12} & 0 & 0 & \tilde{B}_{Q12} \\ \hline \tilde{C}_{r1} & \tilde{C}_{r2} & 0 & \tilde{D}_{rn} & \tilde{D}_{rw1} & \tilde{D}_{rw2} \\ \tilde{C}_{z11} & 0 & \tilde{C}_{Q12} & \tilde{D}_{zn1} & 0 & 0 \\ \tilde{C}_{z21} & \tilde{C}_{z22} & 0 & \tilde{D}_{zn2} & D_{Q21} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{Q12} \\ n \\ w_s \\ w_a \end{bmatrix}$$

Theorem 1 implies  $\|\mathbf{H}_{\text{OF}}\|_{\text{MSP}} < \nu$  iff there exists  $X \in \mathbf{S}_{++}^{\tilde{n}}$ ,  $R \in \mathbf{S}_{++}^{l_r}$ ,  $Z \in \mathbf{S}_{++}^{l_z}$  and  $W \in \mathbf{B}_{++}^{\eta}$  satisfying the LMIs

$$X \succ \tilde{A}X\tilde{A}^T + \tilde{B}_n\tilde{B}_n^T + \tilde{B}_wW\tilde{B}_w^T \quad (\text{D.5a})$$

$$R \succ \tilde{C}_rX\tilde{C}_r^T + \tilde{D}_{rn}\tilde{D}_{rn}^T + \tilde{D}_{rw}W\tilde{D}_{rw}^T \quad (\text{D.5b})$$

$$Z \succ \tilde{C}_zX\tilde{C}_z^T + \tilde{D}_{zn}\tilde{D}_{zn}^T + \tilde{D}_{zw}W\tilde{D}_{zw}^T \quad (\text{D.5c})$$

$$W \succ \Sigma \circ Z \quad (\text{D.5d})$$

with  $\nu^2 > \text{tr}(R)$ . We can construct a feasible  $X \succ 0$  with a structure (D.6)

$$X = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{12}^T & X_{22} & X_{23} \\ 0 & X_{23}^T & X_{33} \end{bmatrix} \quad (\text{D.6})$$

following the partitioning of  $\tilde{A}$ . To do so, consider the recursion

$$\tilde{X}(k+1) = \tilde{A}\tilde{X}(k)\tilde{A}^T + \tilde{B}_w\tilde{W}(k)\tilde{B}_w^T \quad (\text{D.7a})$$

$$\tilde{R}(k) = \tilde{C}_r\tilde{X}(k)\tilde{C}_r^T + \tilde{D}_{rw}\tilde{W}(k)\tilde{D}_{rw}^T \quad (\text{D.7b})$$

$$\tilde{Z}(k) = \tilde{C}_z\tilde{X}(k)\tilde{C}_z^T + \tilde{D}_{zw}\tilde{W}(k)\tilde{D}_{zw}^T \quad (\text{D.7c})$$

$$\tilde{W}(k) = \Sigma \circ \tilde{Z}(k) \quad (\text{D.7d})$$

which is such that  $\tilde{X}(k) \rightarrow 0$ ,  $\tilde{R}(k) \rightarrow 0$ ,  $\tilde{Z}(k) \rightarrow 0$ , and  $\tilde{W}(k) \rightarrow 0$  as  $k \rightarrow \infty$ , where linearity implies the convergence is exponential. Therefore if we let  $P(k) \triangleq \sum_{j=0}^k \tilde{X}(j)$ , then  $P = \lim_{k \rightarrow \infty} P(k)$  exists. Additionally, due to the structure of  $\tilde{A}$  and  $\tilde{B}_w$ , if we choose  $\tilde{X}(0) \succ 0$  with the structure (D.6), then  $\tilde{X}(k)$  has this structure  $\forall k \geq 0$  and hence so does  $P$ .

Now by Corollary 14, the state covariance  $\tilde{X}(k)$  can be expressed

$$\tilde{X}(k+1) = \tilde{A}\tilde{X}(k)\tilde{A}^T + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z \tilde{X}(k) \tilde{C}_z^T \right) \right] \right] \tilde{B}_w^T \quad (\text{D.8a})$$

Using this it is easy to verify that  $P(k)$  satisfies

$$P(k+1) = \tilde{A}P(k)\tilde{A}^T + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z P(k) \tilde{C}_z^T \right) \right] \right] \tilde{B}_w^T + \tilde{X}(0) \quad (\text{D.9})$$

and

$$P = \tilde{A}P\tilde{A}^T + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z P \tilde{C}_z^T \right) \right] \right] \tilde{B}_w^T + \tilde{X}(0) \quad (\text{D.10})$$

Therefore  $\tilde{X}(0) \succ 0$  implies

$$P \succ \tilde{A}P\tilde{A}^T + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z P \tilde{C}_z^T \right) \right] \right] \tilde{B}_w^T \quad (\text{D.11})$$

Again by Corollary 14, the state correlation recursion for  $\mathbf{H}_{\text{OF}} = \mathbf{F}_\ell(\mathbf{T}_{\text{OF}}, \Delta)$  can be expressed

$$X(k+1) = \tilde{A}X(k)\tilde{A}^T + \tilde{B}_n N \tilde{B}_n + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z X(k) \tilde{C}_z^T + \tilde{D}_{zn} N \tilde{D}_{zn}^T \right) \right] \right] \tilde{B}_w^T \quad (\text{D.12a})$$

and for any  $X(0)$ ,  $X(k) \rightarrow \bar{X} \succeq 0$  as  $k \rightarrow \infty$  where

$$\bar{X} = \tilde{A}\bar{X}\tilde{A}^T + \tilde{B}_n N \tilde{B}_n + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z \bar{X} \tilde{C}_z^T + \tilde{D}_{zn} N \tilde{D}_{zn}^T \right) \right] \right] \tilde{B}_w^T \quad (\text{D.13a})$$

and  $\bar{X}$  has the structure in (D.6). To see this, recall that  $\bar{X}$  is unique for all  $X(0) \succeq 0$ , and that for  $X(0) \succeq 0$  with the structure in (D.6),  $X(k)$  will have this structure  $\forall k \geq 0$  and hence so does  $\bar{X}$ .

Therefore if we let  $X = \bar{X} + \alpha P$ , then for some  $\alpha > 0$  we have

$$X \succ \tilde{A}X\tilde{A}^T + \tilde{B}_n \tilde{B}_n + \tilde{B}_w \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z X \tilde{C}_z^T + \tilde{D}_{zn} \tilde{D}_{zn}^T \right) \right] \right] \tilde{B}_w^T \quad (\text{D.14a})$$

$$R \succ \tilde{C}_r X \tilde{C}_r^T + \tilde{D}_{rn} \tilde{D}_{rn} + \tilde{D}_{rw} \left[ \Sigma \circ \left[ \sum_{m=0}^{q-1} D^m \left( \tilde{C}_z X \tilde{C}_z^T + \tilde{D}_{zn} \tilde{D}_{zn}^T \right) \right] \right] \tilde{D}_{rw}^T \quad (\text{D.14b})$$

with  $\nu^2 > \text{tr}(R)$ . We know from the proof of Theorem 1 that for this  $X$  and  $R$  there exist  $Z \in \mathbf{S}_{++}^l$  and  $W \in \mathbf{B}_{++}^\eta$  such that

$$X \succ \tilde{A}X\tilde{A}^T + \tilde{B}_n\tilde{B}_n^T + \tilde{B}_wW\tilde{B}_w^T \quad (\text{D.15a})$$

$$R \succ \tilde{C}_rX\tilde{C}_r^T + \tilde{D}_{rn}\tilde{D}_{rn}^T + \tilde{D}_{rw}W\tilde{D}_{rw}^T \quad (\text{D.15b})$$

$$Z \succ \tilde{C}_zX\tilde{C}_z^T + \tilde{D}_{zn}\tilde{D}_{zn}^T + \tilde{D}_{zw}W\tilde{D}_{zw}^T \quad (\text{D.15c})$$

$$W \succ \Sigma \circ Z \quad (\text{D.15d})$$

and  $\|\mathbf{H}_{\text{OF}}\|_{MSP}^2$  is the infimum of  $\text{tr}(R)$  subject to (D.15a)-(D.15d).

Note the output of  $\mathbf{Q}_{12}$  does not enter the states  $x_1(k)$  and  $x_2(k)$  in (D.4a)-(D.4b) or the performance  $r(k)$  directly, but only  $z_s(k)$  via  $\tilde{C}_{\mathbf{Q}_{12}}$ . Since we restrict  $X$  to the structure in (D.6), the block diagonal elements of (D.15d) are given by

$$W_s \succ \Sigma_s \circ (\tilde{C}_{z_{11}}X_{11}\tilde{C}_{z_{11}}^T + C_{\mathbf{Q}_{12}}X_{33}C_{\mathbf{Q}_{12}}^T + \tilde{D}_{zn_1}\tilde{D}_{zn_1}^T) \quad (\text{D.16a})$$

$$W_a \succ \Sigma_a \circ (\tilde{C}_{z_{21}}X_{11}\tilde{C}_{z_{21}}^T + \tilde{C}_{z_{22}}X_{12}\tilde{C}_{z_{21}}^T + \tilde{C}_{z_{21}}X_{12}\tilde{C}_{z_{22}}^T + \tilde{C}_{z_{22}}X_{22}\tilde{C}_{z_{22}}^T + D_{\mathbf{Q}_{21}}W_sD_{\mathbf{Q}_{21}}^T) \quad (\text{D.16b})$$

Since  $X_{33} \succ 0$ , inequalities (D.15a)-(D.15d) are still satisfied with the same  $X$ ,  $R$ ,  $Z$  and  $W$  if  $C_{\mathbf{Q}_{12}} = 0$ . Therefore this preserves MS stability. In fact, making  $C_{\mathbf{Q}_{12}} = 0$  relaxes these inequalities, and the infimum of  $\text{tr}(R)$  subject to relaxed inequalities cannot be greater.

Finally we simply note that under the condition that  $\mathbf{Q}_{12}$  is strictly proper,  $C_{\mathbf{Q}_{12}} = 0$  forces  $\mathbf{Q}_{12} = 0$ . Therefore if  $\bar{\mathbf{H}}_{\text{OF}} = F_\ell(\mathcal{T}_{\text{OF}}(F, L, L_0, \bar{\mathbf{Q}}), \Delta)$  with  $\bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_{11} & 0 \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$ . Then  $\bar{\mathbf{H}}_{\text{OF}} \in \mathbf{H}_{\text{OF}}$  and  $\|\bar{\mathbf{H}}_{\text{OF}}\|_{MSP} \leq \|\mathbf{H}_{\text{OF}}\|_{MSP}$ .  $\square$

*Proof of Lemma 11.* Recall Notation 4.  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$  and  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$  are given by

$$\bar{L} = -(AX_rC_y^T)(C_yX_rC_y^T + D_nD_n^T + \bar{W}_s)^{-1} \quad (\text{D.17a})$$

$$\bar{L}_0 = -(C_rX_rC_y^T + D_{rn}D_n^T)(C_yX_rC_y^T + D_nD_n^T + \bar{W}_s)^{-1} \quad (\text{D.17b})$$

where  $X_r$  solves the corresponding  $\mathcal{H}_2$  Riccati equation, or equivalently the Lyapunov equation

$$X_r = (A + \bar{L}C_y)X_r(A + \bar{L}C_y)^T + (B_n + \bar{L}D_n)(B_n + \bar{L}D_n)^T + \bar{L}\bar{W}_s\bar{L}^T \quad (\text{D.18})$$

Recalling Notation 3,  $\bar{W}_s$  is obtained from the steady state solution obtained by applying Lemma 1 to the given  $H_{\text{OF}} = F_\ell(\mathsf{T}_{\text{OF}}, \Delta)$  with any realization of  $\mathsf{T}_{\text{OF}}$ . This is because  $\bar{W}$ ,  $\bar{Z}$ , and  $\bar{R}$  are the steady state covariances of input/output signals to  $\mathsf{T}_{\text{OF}}$ , and hence are invariant with respect to the particular state realization. (See Corollary 2.)

Now for any  $F$  such that  $A+B_uF$  is Schur and the subsequent  $Q$  such that  $\mathsf{T}_{\text{OF}} = \mathcal{T}_{\text{OF}}(F, \bar{L}, \bar{L}_0, Q)$ , let  $\mathsf{T}_{\text{OF}}$  have a realization

$$\mathsf{T}_{\text{OF}} : \begin{bmatrix} \hat{x}^+ \\ r \\ z \end{bmatrix} \left[ \begin{array}{c|cc} \hat{A} & \hat{B}_n & \hat{B}_w \\ \hline \hat{C}_r & \hat{D}_{rn} & \hat{D}_{rw} \\ \hat{C}_z & \hat{D}_{zn} & \hat{D}_{zw} \end{array} \right] \begin{bmatrix} \hat{x} \\ n \\ w \end{bmatrix} \quad (\text{D.19})$$

with  $\hat{x} \triangleq \begin{bmatrix} e \\ \hat{x} \\ x_Q \end{bmatrix} \in \mathbf{R}^{n'}$  as in Proposition 11, Figure 6.3. From Lemma 1 we obtain

$$\hat{X} = \hat{A}\hat{X}\hat{A}^T + \hat{B}_n\hat{B}_n^T + \hat{B}_w\bar{W}\hat{B}_w^T \quad (\text{D.20a})$$

$$\bar{R} = \hat{C}_r\hat{X}\hat{C}_r^T + \hat{D}_{rn}\hat{D}_{rn}^T + \hat{D}_{rw}\bar{W}\hat{D}_{rw}^T \quad (\text{D.20b})$$

$$\bar{W} = \Sigma \circ (\hat{C}_z\hat{X}\hat{C}_z^T + \hat{D}_{zn}\hat{D}_{zn}^T + \hat{D}_{zw}\bar{W}\hat{D}_{zw}^T) \quad (\text{D.20c})$$

Partition  $\hat{X}$  in (D.20a) as

$$\hat{X} = \begin{bmatrix} \hat{X}_e & \hat{X}_{e\hat{x}} & \hat{X}_{ex_Q} \\ (\hat{X}_{e\hat{x}})^T & \hat{X}_{\hat{x}} & \hat{X}_{\hat{x}x_Q} \\ (\hat{X}_{ex_Q})^T & (\hat{X}_{\hat{x}x_Q})^T & \hat{X}_Q \end{bmatrix} \quad (\text{D.21})$$

Following some tedious algebra–working out the structure of  $\dot{A}$ ,  $\dot{B}_n$ , and  $\dot{B}_w$ , then expanding (D.20a) we can express the  $\dot{X}_e$ ,  $\dot{X}_{e\hat{x}}$ , and  $\dot{X}_{ex_Q}$  blocks of  $\dot{X}$  in Equation (D.21) as

$$\begin{aligned} \dot{X}_e &= (A + \bar{L}C_y)\dot{X}_e(A + \bar{L}C_y)^T \\ &\quad + (B_n + \bar{L}D_n)(B_n + \bar{L}D_n)^T + \bar{L}\bar{W}_s\bar{L}^T \end{aligned} \quad (\text{D.22a})$$

$$\begin{aligned} \dot{X}_{e\hat{x}} &= (A + \bar{L}C_y)\dot{X}_{e\hat{x}}(A + B_uF)^T + (A + \bar{L}C_y)\dot{X}_{ex_Q}(B_uC_{Q_2})^T \\ &\quad + (A\dot{X}_eC_y^T)(B_uD_{Q_{21}})^T \\ &\quad + \bar{L}(C_y\dot{X}_eC_y^T + D_nD_n^T + \bar{W}_s)(B_uD_{Q_{21}})^T \\ &\quad - (A\dot{X}_eC_y^T)\bar{L}^T \\ &\quad - \bar{L}(C_y\dot{X}_eC_y^T + D_nD_n^T + \bar{W}_s)\bar{L}^T \end{aligned} \quad (\text{D.22b})$$

$$\begin{aligned} \dot{X}_{ex_Q} &= (A + \bar{L}C_y)\dot{X}_{ex_Q}(A_Q)^T + (A\dot{X}_eC_y^T)B_{Q_1}^T \\ &\quad + \bar{L}(C_y\dot{X}_eC_y^T + D_nD_n^T + \bar{W}_s)B_{Q_1}^T \end{aligned} \quad (\text{D.22c})$$

Note that  $\dot{X}_e$  above solves an identical Lyapunov equation as  $X_r$  in Equation (D.18), and  $(A + \bar{L}C_y)$  is Schur. By Proposition 14 the solution is unique. Therefore  $\dot{X}_e = X_r$  and substituting  $\bar{L}$  from Equation (D.17a) into  $\dot{X}_{e\hat{x}}$  and  $\dot{X}_{ex_Q}$  and simplifying gives

$$\begin{aligned} \dot{X}_{e\hat{x}} &= (A + \bar{L}C_y)\dot{X}_{e\hat{x}}(A + B_uF)^T + (A + \bar{L}C_y)\dot{X}_{ex_Q}(B_uC_{Q_2})^T \\ \dot{X}_{ex_Q} &= (A + \bar{L}C_y)\dot{X}_{ex_Q}(A_Q)^T \end{aligned}$$

Because  $(A + \bar{L}C_y)$ ,  $(A + B_uF)$ , and  $A_Q$  are Schur matrices, Proposition 13 (Appendix A) implies  $\dot{X}_{ex_Q} = 0$  and  $\dot{X}_{e\hat{x}} = 0$ .

Now focusing on the MS performance, we can separate  $\bar{R}$  in (D.20b) as  $\bar{R} \triangleq \bar{R}_e + \bar{R}_{e\hat{x}} + (\bar{R}_{e\hat{x}})^T + \bar{R}_{\hat{x}}$ . Using  $\dot{X}_{e\hat{x}} = 0$  and  $\dot{X}_{ex_Q} = 0$  we expand each as

$$\begin{aligned} \bar{R}_e &= (C_r + \bar{L}_0 C_y) \dot{X}_e (C_r + \bar{L}_0 C_y)^T \\ &\quad + (D_{rn} + \bar{L}_0 D_n) (D_{rn} + \bar{L}_0 D_n)^T + \bar{L}_0 \bar{W}_s \bar{L}_0^T \end{aligned} \quad (\text{D.24a})$$

$$\begin{aligned} \bar{R}_{e\hat{x}} &= \left[ C_r \dot{X}_e C_y^T + D_{rn} D_n^T + \bar{L}_0 (C_y \dot{X}_e C_y^T + D_n D_n^T + \bar{W}_s) \right] D_{Q_{21}}^T D_r^T \\ &\quad - (C_r \dot{X}_e C_y^T + D_{rn} D_n^T) \bar{L}_0 - \bar{L}_0 (C_y \dot{X}_e C_y^T + D_n D_n^T + \bar{W}_s) \bar{L}_0^T \\ \bar{R}_{\hat{x}} &= \begin{bmatrix} C_r + D_r F & D_r C_{Q_2} \end{bmatrix} \begin{bmatrix} \dot{X}_{\hat{x}} & \dot{X}_{\hat{x}x_Q} \\ \dot{X}_{\hat{x}x_Q}^T & \dot{X}_{\hat{x}} \end{bmatrix} \begin{bmatrix} C_r^T + F^T D_r^T \\ C_{Q_2}^T D_r^T \end{bmatrix} \\ &\quad + (D_r D_{Q_{21}} - \bar{L}_0) S (D_r D_{Q_{21}} - \bar{L}_0)^T + D_{rw} \bar{W}_a D_{rw}^T \end{aligned} \quad (\text{D.24b})$$

with

$$S \triangleq C_y \dot{X}_e C_y^T + D_n D_n^T + \bar{W}_s \quad (\text{D.24c})$$

Recalling  $X_r = \dot{X}_e$  and substituting  $\bar{L}_0$  from (D.17b) into  $\bar{R}_{e\hat{x}}$  and simplifying we get  $\bar{R}_{e\hat{x}} = 0$  which means

$$\bar{R} = \bar{R}_e + \bar{R}_{\hat{x}} \Rightarrow \text{tr}(\bar{R}) = \text{tr}(\bar{R}_e) + \text{tr}(\bar{R}_{\hat{x}}) \quad (\text{D.25})$$

where  $\text{tr}(\bar{R}_e)$  and  $\text{tr}(\bar{R}_{\hat{x}})$  are the squared MS performance values for the  $r_e(k)$  and  $r_{\hat{x}}(k)$  outputs in Figure 6.3. Equivalently, referring to (D.24a)-(D.24c),  $\|\mathbf{H}_{\text{OF}}\|_{\text{MSP}}^2 = \|\mathbf{H}_{\text{FC}}\|_{\text{MSP}}^2 + \|\mathbf{H}_{\text{DF}} \bar{S}^{\frac{1}{2}}\|_{\text{MSP}}^2$   $\square$

The proof of Lemma 11 implies the following corollary which will be useful in the proof of Corollary 13 to follow.

**Corollary 15.** *Given any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}_{\text{OF}}''$  with  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{\text{OF}})$ ,  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$ ,  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$ , and  $\bar{S} = \mathcal{S}^\perp(\bar{W}_s)$ , let  $\mathbf{H}_{\text{FC}} = \mathcal{H}_{\text{FC}}(\bar{L}, \bar{L}_0, \mathbf{Q}_{11}, \Delta_s)$  and define the stable system*

$$\hat{\mathbf{G}}_{\text{FC}}^{\bar{W}}: \begin{bmatrix} \bar{e}^+ \\ \bar{r} \\ s \end{bmatrix} = \begin{bmatrix} A + \bar{L} C_y & B_n + \bar{L} D_n & \bar{L} \bar{W}_s^{\frac{1}{2}} \\ C_r + \bar{L}_0 C_y & D_{rn} + \bar{L}_0 D_n & \bar{L}_0 \bar{W}_s^{\frac{1}{2}} \\ C_y & D_n & \bar{W}_s^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \bar{e} \\ \bar{n}_1 \\ \bar{n}_2 \end{bmatrix} \quad (\text{D.26})$$

and  $S = \lim_{k \rightarrow \infty} \mathbb{E}(s(k) s^T(k))$ . Then  $\bar{S} = S$ , and if we let  $\mathbf{T}_{\text{FC}}^{\bar{W}}$  be the part of  $\hat{\mathbf{G}}_{\text{FC}}^{\bar{W}}$  with output  $\bar{r}$ ,

$$\|\mathbf{H}_{\text{FC}}\|_{\text{MSP}} = \|\mathbf{T}_{\text{FC}}^{\bar{W}}\|_2.$$

*Proof of Corollary 13.* Any  $\mathbf{H}_{\text{OF}} \in \mathbf{H}_{\text{OF}}''$  has a realization which is the interconnection of  $\mathbf{H}_{\text{FC}} = \mathcal{H}_{\text{FC}}(\bar{L}, \bar{L}_0, \mathbf{Q}_{11}, \Delta_s)$  and  $\mathbf{H}_{\text{DF}} = \mathcal{H}_{\text{DF}}(\bar{L}, \bar{L}_0, F^*, [F'_0 \ 0], \Delta_a)$  where  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{\text{OF}})$ ,  $\bar{S} = \mathcal{S}^\perp(\bar{W}_s)$ ,  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$  and  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$ , and  $F'_0 = \mathcal{F}(\bar{W}_s)$ .

Using these quantities, define the following systems.

$$\hat{\mathbf{G}}_{\text{FC}}^{\bar{W}}: \begin{bmatrix} \bar{e}^+ \\ \bar{r}_{\text{FC}} \\ s \end{bmatrix} = \left[ \begin{array}{c|cc} A + \bar{L}C_y & B_n + \bar{L}D_n & \bar{L}\bar{W}_s^{\frac{1}{2}} \\ \hline C_r + \bar{L}_0C_y & D_{rn} + \bar{L}_0D_n & \bar{L}_0\bar{W}_s^{\frac{1}{2}} \\ C_y & D_n & \bar{W}_s^{\frac{1}{2}} \end{array} \right] \begin{bmatrix} \bar{e} \\ \bar{n}_1 \\ \bar{n}_2 \end{bmatrix} \quad (\text{D.27})$$

$$\mathbf{G}_{\text{DF}}^{\bar{W}}: \begin{bmatrix} \bar{x}^+ \\ \bar{r}_{\text{DF}_1} \\ \bar{r}_{\text{DF}_2} \\ \bar{y} \end{bmatrix} = \left[ \begin{array}{c|cc} A & -\bar{L} & B_u \\ \hline C_r & -\bar{L}_0 & D_r \\ 0 & 0 & \bar{V}^{\frac{1}{2}} \\ C_y & I & 0 \end{array} \right] \begin{bmatrix} x_G \\ s \\ u \end{bmatrix} \quad (\text{D.28})$$

where  $\bar{V} = \Sigma \circ (B_w^T X_{\text{DF}} B_w + D_{rw}^T D_{rw})$ .

It is easy to verify that  $\mathbf{G}_{\text{OF}}^{\bar{W}}$  in (6.17) is equal to the interconnection of  $\hat{\mathbf{G}}_{\text{FC}}^{\bar{W}}$  in (D.27) and  $\mathbf{G}_{\text{DF}}^{\bar{W}}$  in (D.28). These two systems have the structure of the classical FC/DF split for  $\mathbf{G}_{\text{OF}}^{\bar{W}}$ . Moreover  $\bar{L}$  and  $\bar{L}_0$  are the optimal  $\mathcal{H}_2$  FC gains for this problem, and since (D.28) is an instance of (5.22), Lemma 5 implies that  $F^*$  and  $F'_0$  are the optimal  $\mathcal{H}_2$  DF gains for this problem. Therefore if we let  $\mathbf{T}_{\text{FC}}^{\bar{W}}$  be the part of  $\hat{\mathbf{G}}_{\text{FC}}^{\bar{W}}$  with output  $\bar{r}_{\text{FC}}$ ,  $S = \lim_{k \rightarrow \infty} \mathbf{E}(s(k)s^T(k))$ ,  $\mathbf{T}_{\text{DF}}^{\bar{W}}$  be the resulting DF closed loop, and  $\mathbf{T}_{\text{OF}}^{\bar{W}}$  be the overall closed loop, then  $\|\mathbf{T}_{\text{OF}}^{\bar{W}}\|_2^2 = \|\mathbf{T}_{\text{FC}}^{\bar{W}}\|_2^2 + \|\mathbf{T}_{\text{DF}}^{\bar{W}} S^{\frac{1}{2}}\|_2^2$ .

Finally, we know that  $\bar{S} = S$  and  $\|\mathbf{T}_{\text{FC}}^{\bar{W}}\|_2 = \|\mathbf{H}_{\text{FC}}\|_{\text{MSP}}$  by Corollary 15, and  $\|\mathbf{T}_{\text{DF}}^{\bar{W}} S^{\frac{1}{2}}\|_2 = \|\mathbf{H}_{\text{DF}} \bar{S}^{\frac{1}{2}}\|_{\text{MSP}}$  by Lemma 5. That is,  $\|\mathbf{T}_{\text{OF}}^{\bar{W}}\|_2^2 = \|\mathbf{H}_{\text{FC}}\|_{\text{MSP}}^2 + \|\mathbf{H}_{\text{DF}} \bar{S}^{\frac{1}{2}}\|_{\text{MSP}}^2 = \|\mathbf{H}_{\text{OF}}\|_{\text{MSP}}^2$ .  $\square$

*Proof of Theorem 2.* We can verify that the controller  $\mathbf{K}_{\text{OF}}^*$  as given in (6.20) is MS stabilizing by noting that the resulting closed loop  $\mathbf{H}_{\text{OF}}^*$  is equal to the interconnection of  $\mathbf{H}_{\text{FC}}^* = \mathcal{H}_{\text{FC}}(L^*, L_0^*, 0, \Delta_s)$  and  $\mathbf{H}_{\text{DF}}^* = \mathcal{H}_{\text{DF}}(L^*, L_0^*, F^*, [F_0^* \ 0], \Delta_a)$  in Figure 6.4 where both  $\mathbf{H}_{\text{FC}}^*$  and  $\mathbf{H}_{\text{DF}}^* = \mathcal{H}_{\text{DF}}(L^*, L_0^*, F^*, [F_0^* \ 0], \Delta_a)$  are MS stable. Moreover, if we let  $W_s^* = \mathcal{W}(\mathbf{H}_{\text{OF}}^*)$  then by Lemma 9 and Corollary 15,  $L^* = \mathcal{L}^\perp(W_s^*)$ ,  $L_0^* = \mathcal{L}_0^\perp(W_s^*)$ ,  $S_{\text{FC}} = \mathcal{S}^\perp(W_s^*)$ , meaning by Lemma 11 we have

$$\|\mathbf{H}_{\text{OF}}^*\|_{\text{MSP}}^2 = \|\mathbf{H}_{\text{FC}}^*\|_{\text{MSP}}^2 + \|\mathbf{H}_{\text{DF}}^* S_{\text{FC}}^{\frac{1}{2}}\|_{\text{MSP}}^2$$

Moreover, with  $\bar{L} = L^*$  and  $\bar{L}_0 = L_0^*$  we have  $F_0^* = F'_0 = \mathcal{F}(\bar{W}_s)$ , and  $\mathbf{H}_{\text{OF}}^* \in \mathbf{H}_{\text{OF}}''$ .

Now consider any other  $\bar{\mathbf{H}}_{\text{OF}} \in \mathbf{H}_{\text{OF}}''$  and let  $\bar{W}_s = \mathcal{W}(\mathbf{H}_{\text{OF}})$ ,  $\bar{L} = \mathcal{L}^\perp(\bar{W}_s)$ ,  $\bar{L}_0 = \mathcal{L}_0^\perp(\bar{W}_s)$ . Recalling the proof of Lemma 11, we have that substituting  $\bar{L}$  into (D.22a) gives us

$$\dot{X}_e = A\dot{X}_e A^T + B_n B_n^T - (A\dot{X}_e C_y^T)(C_y \dot{X}_e C_y^T + D_n D_n^T + \bar{W}_s)^{-1}(C_y \dot{X}_e A^T)$$

where

$$\bar{W}_s = \Sigma_s \circ (C_z \dot{X}_e C_z^T + D_{zn} D_{zn}^T + C_{Q_1} \dot{X}_Q C_{Q_1}^T) \quad (\text{D.29})$$

and<sup>2</sup>

$C_{Q_1} \dot{X}_Q C_{Q_1}^T \neq 0 \Leftrightarrow Q_{11} \neq 0$ . Similarly, for  $L^*$  in (6.18d) we have

$$X_e^* = A X_e^* A^T + B_n B_n^T - (A X_e^* C_y^T)(C_y X_e^* C_y^T + D_n D_n^T + \bar{W}_s^*)^{-1}(C_y X_e^* A^T)$$

where

$$W_s^* = \Sigma_s \circ (C_z X_e^* C_z^T + D_{zn} D_{zn}^T) \quad (\text{D.30})$$

Note that since  $C_{Q_1} \dot{X}_Q C_{Q_1}^T \succeq 0$  the following inequality is satisfied by our quantities.

$$\begin{bmatrix} B_n B_n^T & 0 \\ 0 & D_n D_n^T + \Sigma_s \circ (D_{zn} D_{zn}^T + C_{Q_1} \dot{X}_Q C_{Q_1}^T) \end{bmatrix} \succeq \begin{bmatrix} B_n B_n^T & 0 \\ 0 & D_n D_n^T + \Sigma_s \circ (D_{zn} D_{zn}^T) \end{bmatrix} \quad (\text{D.31})$$

Inequality (D.31) means we meet the conditions of Lemma 5.4 in [67], which directly tells us that  $\dot{X}_e \succeq X_e^*$ . Since  $C_{Q_1} \dot{X}_Q C_{Q_1}^T \succeq 0$  this means  $\bar{W}_s \succeq W_s^*$ .

By Corollary 13,  $\|\bar{\mathbf{H}}_{\text{OF}}\|_{\text{MSP}}$  is equal to the optimal  $\mathcal{H}_2$  performance given a plant (6.17) subject to an exogenous noise with covariance  $\bar{W}_s$ , and  $\|\mathbf{H}_{\text{OF}}^*\|_{\text{MSP}}$  is equivalent to the optimal  $\mathcal{H}_2$  performance given the same plant subject to an exogenous noise with covariance  $W_s^*$ . We have just shown that  $W_s^* \preceq \bar{W}_s$ . We conclude that  $\|\mathbf{H}_{\text{OF}}^*\|_{\text{MSP}} \leq \|\bar{\mathbf{H}}_{\text{OF}}\|_{\text{MSP}}$  for any  $\bar{\mathbf{H}}_{\text{OF}} \in \mathbf{H}_{\text{OF}}''$ , i.e.  $\nu_{\text{OF}}^* = \|\mathbf{H}_{\text{OF}}^*\|_{\text{MSP}}$ .  $\square$

<sup>2</sup>Using Proposition 14,  $C_{Q_1} \dot{X}_Q C_{Q_1}^T = C_{Q_1} B_{Q_1} S B_{Q_1}^T C_{Q_1}^T + C_{Q_1} A_Q B_{Q_1} S B_{Q_1}^T A_Q^T C_{Q_1}^T + \dots$  which since  $S \succ 0$  is zero if and only if  $Q_{11} = 0$  (i.e. has zero impulse response).



## APPENDIX E. NOTATION

### Sets and Vector Spaces

$\emptyset$	Empty set
$\mathbf{N}$	Natural numbers (nonnegative integers)
$\mathbf{R}$	Real numbers
$\mathbf{R}^n$	Real $n$ -vectors ( $n \times 1$ matrices)
$\mathbf{R}^{n \times m}$	Real $n \times m$ matrices
$\mathbf{S}^n$	Symmetric $n \times n$ matrices
$\mathbf{S}_+^n$	Symmetric positive semidefinite $n \times n$ matrices
$\mathbf{S}_{++}^n$	Symmetric positive definite $n \times n$ matrices
$\mathbf{D}^n$	$n \times n$ diagonal matrices
$\mathbf{D}_+^n$	$n \times n$ diagonal positive semidefinite matrices
$\mathbf{D}_{++}^n$	$n \times n$ diagonal positive definite matrices
$\mathbf{B}^\eta$	Block diagonal matrices made of symmetric $\eta_i \times \eta_i$ blocks
$\mathbf{B}_+^\eta$	Block diagonal positive semidefinite matrices made of symmetric $\eta_i \times \eta_i$ blocks
$\mathbf{B}_{++}^\eta$	Block diagonal positive definite matrices made of symmetric $\eta_i \times \eta_i$ blocks
$a_i$	The $i$ th element of some indexed set
$\{a_i\}$	The set of some elements $a_i$ for $i = 1, \dots$

### Vectors, matrices, and linear operators

$v_i$	If $v$ is a vector, the $i$ th element of the vector $v$
$M_{ij}$	Element in the $i$ th row and $j$ th column of the matrix $M$
$I_n$	$n \times n$ identity matrix
$I$	Identity matrix of whatever size is appropriate in context

$\mathbf{1}_{n \times m}$	$n \times m$ matrix of ones
$\mathbf{1}$	Vector of ones of whatever size appropriate in context
$\mathbf{0}$	Matrix of zeros of whatever size is appropriate in context
$X^T$	Transpose of the matrix $X$
$X^{-1}$	Inverse of the matrix $X$
$X^\dagger$	Moore-Penrose pseudo-inverse of the matrix $X$
$\text{tr}(X)$	Trace of the matrix $X$
$\text{diag}(X, Y, \dots)$	Block diagonal matrix with diagonal blocks $X, Y, \dots$
$X \circ Y$	The Hadamard (element-by-element) product of the matrices $X$ and $Y$
$X \otimes Y$	The Kronecker product of the matrices $X$ and $Y$
$\text{vec}(X)$	The vectorization of the matrix $X$ , i.e. all columns of $X$ stacked into a vector
$X \succ 0$	$X$ is a symmetric positive definite matrix
$X \succeq 0$	$X$ is a symmetric positive semidefinite matrix
$X \succ Y$	$X - Y$ is a symmetric positive definite matrix
$X \succeq Y$	$X - Y$ is a symmetric positive semidefinite matrix
$X^{\frac{1}{2}}$	For $X \succeq 0$ , the unique symmetric matrix such that $X^{\frac{1}{2}} \succeq 0$ and $(X^{\frac{1}{2}})^2 = X$
$X^{\frac{1}{2}}$	For $X \succ 0$ , the unique symmetric matrix such that $X^{\frac{1}{2}} \succ 0$ and $(X^{\frac{1}{2}})^2 = X$
$T^n(X)$	Recursive application of $T : \mathbf{V} \mapsto \mathbf{V}$ for $X \in \mathbf{V}$ , e.g. $T^2(X) = T(T(X))$
$T^0(X)$	Identity application of $T : \mathbf{V} \mapsto \mathbf{V}$ for $X \in \mathbf{V}$ , i.e. $T^0(X) = X$
$\rho(T)$	The spectral radius of the linear operator $T$

### Linear systems and discrete time indexing

$G$	A linear system
$G^T$	The dual linear system of $G$ (See (1.1) and (1.2) page 7.)
$F_\ell(G, K)$	The lower linear fractional transformation (LFT) of $G$ and $K$ (See e.g. [61, 62])
$\text{Star}(G, J)$	The star product of $G$ and $J$ (See e.g. [61])
$\ T\ _2$	The $\mathcal{H}_2$ norm of the LTI system $T$
$\ H\ _{MSP}$	The mean-square performance norm of the LTV system $H$ (Definition 3)

$X(k)$	The quantity $X$ at time $k$
$x^+$	Shorthand for $x(k + 1)$ within linear system state space realizations
$A$ is Schur	For a square matrix $A$ , all eigenvalues have magnitude less than one

### Probability

$E(X)$	Expected value of the quantity $X$
$\text{var}(x)$	Variance of the scalar random variable $x$
$\text{cov}(v)$	Covariance matrix of the random vector $v$
$\text{Pr}(S)$	The probability of event $S$

### Acronyms and abbreviations

MS	Mean-square
LTI	Linear time-invariant
LTV	Linear time-varying
FDLTI	Finite-dimensional linear time-invariant
LMI	Linear matrix inequality
i.i.d.	Independent and identically distributed

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